Consiglio Nazionale delle Ricerche
Istituto di Matematica Applicata
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REPORT SERIES

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15-03

IMATI REPORT Series

Nr. 15-03 – December 2015

IMATI Report Series Nr. 15-03 – 11th December 2015

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On quasi-interpolation operators in spline spaces

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December 31, 2015

Abstract

We propose the construction of a class of L^2 stable quasi-interpolation operators onto the space of splines on tensor-product meshes, in any space dimension. The estimate we propose is robust with respect to knot repetition and to knot "vicinity" (up to $p+1$ knots), so it applies to the most general scenario in which the B-spline functions are known to be well defined.

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1 Introduction

The use of splines as a tool for the numerical discretization of partial differential equations is experiencing a very fast spreading thanks to the advent of isogeometric analysis [13], [8]. Besides the many engineering applications that are object of study within the isogeometric framework, there is also a renewed attention towards the development of theoretical tools that may provide a clear mathematical understanding and solid groundings for isogeometric methods.

A state-of-the-art review on the existing mathematical results can be found in the review paper [3] published in 2014. Indeed, several results exists today starting from approximation properties, to wellposedness for various classes of spline discretizations and problems showing that splines, and the isogeometric framework, can be suitably used in the numerical analysis for a variety of PDEs (elliptic, saddle points, Hodge laplacian, etc). In this paper, we focus on a rather fundamental question that is the approximation properties and the techniques to study them in the most general setting of interests. In [3] and in all the literature until now, approximation properties for splines are analysed under the assumption of local quasi-uniform meshes (see Assumption 1 below), possibly in presence of knot repetition. These results are surely useful but fail to analyse the approximation error in the most general framework: indeed, the spline basis remain stable when up to $p+1$ knots are made closer and closer to each-other (up to becoming coincident) while the interpolation operators proposed become unstable when knot spans collapse to zero.

In the present paper, we fill this gap and we put ourselves in the most general situation. Instead of considering one single choice of interpolation operator, we consider an entire class of operators, mostly inspired by [15] and we analyse there approximation properties under the milder Assumption 2 (see below).

Quasi-interpolants in spline spaces are usually defined as linear combination of locally supported functions $\beta \in \mathcal{B}$ that form a convex partition of unity, namely

$$P(f) = \sum_{\beta \in \mathcal{B}} \lambda_\beta(f)\beta ,$$

where the linear functionals $\lambda_\beta(f)$ may be defined in different ways, by e.g. taking into account the evaluation of the function f, or even related integral/derivative information, at certain points or in regions included in (or close to) the support of β, see for example [10, 15, 16, 18]. The use of spline-based quasi-interpolation schemes is an established technique for the design of effective and reliable approximation algorithms.

In this paper we derive an approximation method in terms of local L^2 projection by exploiting the local stability of the univariate B-spline basis and its tensor-product extension. The stability and approximation properties of the corresponding quasi-intepolation operators are presented. The analysis includes the discussion of mild assumptions on the admissible mesh configuration to be considered.

The remaining of the paper is organised as follows. Section 2 provides a brief overview of isogeometric analysis and introduces some basic notation. In Section 3 we state the assumptions on the meshes that we consider. We analyse the stability properties of the B-spline basis in Section 4 through some estimations for the inverse of the local Bézier extraction operator. The local approximation method is then presented in Section 5, while Section 6 defines the locally supported dual basis. Finally, the properties of the quasi-interpolation operator are discussed in Section 7.

2 Motivation: the isogeometric setting

One of the main motivation of our work is to provide mathematical foundation of isogeometric methods and a rigorous understanding of the properties splines have in practice. To this aim, this section is meant to introduce splines and shortly discuss the isogeometric setting where they are meant to be used.

2.1 Univariate and tensor-product B-splines

Let $\Xi := \{\xi_j\}_{j=1}^{n+p+1}$ be a p-open knot (ordered) vector such that

$$0 = \xi_1 = \cdots = \xi_{p+1} < \xi_{p+2} \leq \cdots \leq \xi_n < \xi_{n+1} = \cdots = \xi_{n+p+1} = 1,$$

where the two positive integers p and n denote a given polynomial degree, and the corresponding number of B-splines defined over the considered knot sequence, respectively. We also introduce the vector $Z := \{\zeta_j\}_{j=1}^{\tilde{n}}$ of knots without repetitions, and denote by m_j the multiplicity of the breakpoint ζ_j, such that

$$\Xi = \{\underbrace{\zeta_1, \dots, \zeta_1}_{m_1 \text{ times}}, \underbrace{\zeta_2, \dots, \zeta_2}_{m_2 \text{ times}}, \dots \underbrace{\zeta_{\tilde{n}}, \dots, \zeta_{\tilde{n}}}_{m_{\tilde{n}} \text{ times}}\},$$

with $\sum_{i=1}^{\tilde{n}} m_i = n + p + 1$. Note that the two extreme knots are repeated $p+1$ times, i.e., $m_1 = m_{\tilde{n}} = p + 1$. We assume that an internal knot can be repeated at most $p+1$ times, i.e., $m_j \leq p + 1$, for $j = 2, \dots, \tilde{n} - 1$.

Let $\{\beta_1, \beta_2, \dots, \beta_n\}$ be the univariate B-spline basis of degree p associated to the knot vector Ξ, see e.g., [9, 19]. Each B-spline is a piecewise polynomial of degree p on the subdivision $\{\zeta_1, \dots, \zeta_{\tilde{n}}\}$ and it has $p - m_j$ continuous derivatives at the breakpoint ζ_j. We remark that B-splines are non-negative, locally supported, and form a convex partition of unity, namely

$$\beta_j \geq 0, \qquad \operatorname{supp} \beta_j = [\xi_j, \xi_{j+p+1}], \qquad \sum_{i=1}^{n} \beta_i(x) = 1 \quad \forall x \in (0, 1).$$

Let $\mathcal{I}$ be the univariate mesh defined by

$$\mathcal{I} := \{[\zeta_j, \zeta_{j+1}] \,|\, j = 1, \dots, \tilde{n} - 1\}.$$

For each $I = [\zeta_j, \zeta_{j+1}] \in \mathcal{I}$ there exists a unique $k = \sum_{i=1}^{j} m_j$ such that $I = [\xi_k, \xi_{k+1}]$ and $\xi_k \neq \xi_{k+1}$. The union of the supports of the B-splines acting on I identifies the *support extension* $\tilde{I}$, namely

$$\tilde{I} := [\xi_{k-p}, \xi_{k+p+1}], \tag{1}$$

Moreover, we define

$$\hat{I} := [\xi_{k-p+1}, \xi_{k+p}]. \tag{2}$$

An example of quadratic B-splines constructed from the open knot vector

$$\Xi = \{0, 0, 0, 1/5, 2/5, 3/5, 3/5, 4/5, 1, 1, 1\}$$

is presented in Figure 1. Notice that, since the knot $\xi_6 = \xi_7 = \zeta_4 = 3/5$ has multiplicity $m_4 = 2$, the fourth, fifth and sixth functions are only continuous at that point.

In order to define a tensor-product d-variate spline space on $\hat{\Omega} := [0, 1]^d \subset \mathbb{R}^d$, let $\mathbf{p} := (p_1, p_2, \dots, p_d)$ be the set of polynomial degrees with respect to each coordinate direction. For $i = 1, 2, \dots, d$, let $\Xi_i := \{\xi_j^{(i)}\}_{j=1}^{n_i+p_i+1}$ be a p_i-open knot vector such that

$$0 = \xi_1^{(i)} = \dots = \xi_{p_i+1}^{(i)} < \xi_{p_i+2}^{(i)} \leq \dots \leq \xi_{n_i}^{(i)} < \xi_{n_i+1}^{(i)} = \dots = \xi_{n_i+p_i+1}^{(i)} = 1,$$

where the two extreme knots are repeated $p_i + 1$ times and any internal knot can be repeated at most $p_i + 1$ times. We denote by $\mathbb{V}$ the tensor-product spline space spanned by the B-spline basis $\mathcal{B}$ defined as the tensor-product of the univariate B-spline bases $\mathcal{B}_1, \dots, \mathcal{B}_d$. Let $\mathcal{Q}$ be tensor-product mesh consisting of the elements $Q = I_1 \times \dots \times I_d$, where I_i is an element (closed interval) of the i-th univariate mesh, for $i = 1, \dots, d$.

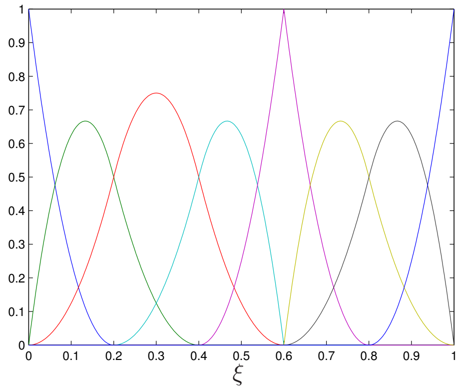

Figure 1: Quadratic B-splines basis functions constructed from the open knot vector $\Xi = \{0, 0, 0, 1/5, 2/5, 3/5, 3/5, 4/5, 1, 1, 1\}$.

2.2 The geometric map and isogeometric refinements

In isogeometric analysis, the physical domain Ω is parametrized by the map $\mathbf{F} : \hat{\Omega} \to \Omega$ given by

$$\mathbf{x} = \mathbf{F}(\hat{\mathbf{x}}), \qquad \hat{\mathbf{x}} \in \hat{\Omega},$$

where $\mathbf{F}$ is a linear combination of the set of B-splines (or their rational extension) defined on an initial, usually coarse, tensor-product grid $\mathcal{Q}_0$. The map $\mathbf{F}$ is assumed to be invertible, with smooth inverse, on each mesh element.

The approximation space on Ω is given by $\text{span}\{\beta \circ \mathbf{F}^{-1}\}_{\beta \in \mathcal{B}}$ as the push-forward of the spline space on $\hat{\Omega}$ and its approximation properties influence the accuracy of the corresponding isogeometric method. Three refinement possibilities are available and are usually indicated as $h-$refinement (mesh refinement), $p-$refinement (degree raising) and $k-$refinement (mesh refinement and degree raising) [1, 2, 6, 11, 13]. The different kinds of refinements are all constructed by applying the standard knot insertion and degree elevation algorithms, see [9, 13]. By exploiting these refinement procedures, refined approximation spaces with various mesh-size, order, and global regularity may be obtained from the initial spline space.

3 Assumptions

The main goal of this article is to build a quasi-interpolant operator for tensor-product spline spaces, assuming that the underlying univariate meshes with respect to the different coordinate directions satisfy one of the following assumptions.

Assumption 1 (Local quasi-uniformity)**.** *There exists a constant* $\theta > 0$ *such that*

$$\theta^{-1} \leq \frac{\zeta_j - \zeta_{j-1}}{\zeta_{j+1} - \zeta_j} \leq \theta, \qquad \forall\, j = 2, \ldots, \tilde{n} - 1.$$

Assumption 2. *There exists a constant* $\theta > 0$ *such that for every* $I \in \mathcal{I}$ *and* $1 \leq j_1, j_2 \leq n$,

$$\theta^{-1} \leq \frac{|\,\text{supp}\, \beta_{j_1}|}{|\,\text{supp}\, \beta_{j_2}|} \leq \theta.$$

whenever $\operatorname{supp}\beta_{j_1} \cap \operatorname{supp}\beta_{j_2} \supset I$.

Remark 3.1. Assumption 2 holds if and only if there exists a constant[1] $C_2 > 0$ such that

$$\frac{|\tilde{I}|}{|\operatorname{supp}\beta_j|} \leq C_2,$$

for all $I \in \mathcal{I}$ and $1 \leq j \leq n$ such that $I \subset \operatorname{supp}\beta_j$.

Remark 3.2. Assumption 1 implies Assumption 2. On the other hand, Assumption 2 allows the shrinking of (at most) $p+1$ knots and thus, it is weaker than Assumption 1. As an example we can consider $p = 2$ and

$$\Xi := \{0, 0, 0, 1/2 - \varepsilon, 1/2 + \varepsilon, 1, 1, 1\},$$

or

$$\Xi := \{0, 0, 0, 1/2 - \varepsilon, 1/2, 1/2 + \varepsilon, 1, 1, 1\},$$

for $0 < \varepsilon < \frac{1}{4}$. In this case, Assumption 2 holds but Assumption 1 does not, since θ would depend on ε in Assumption 1.

4 Some results in spline spaces

In this section, we introduce bounds for the operator performing the change of basis from univariate B-splines restricted to the single knot span to Bernstein polynomials. This operator is commonly known as Bézier extraction operator. We extend such bounds for the tensor-product case and then, we analyse the local stability of the B-spline basis.

4.1 The inverse of the local Bézier extraction operator

The Bernstein polynomials of degree p on the knot interval $I = [\xi_k, \xi_{k+1}] \in \mathcal{I}$ are defined by

$$B_j^I(x) := \begin{pmatrix} p \\ j-1 \end{pmatrix} \left(\frac{x - \xi_k}{\xi_{k+1} - \xi_k}\right)^{j-1} \left(\frac{\xi_{k+1} - x}{\xi_{k+1} - \xi_k}\right)^{p-j+1}, \qquad j = 1, \ldots, p+1.$$

The set $\mathbb{B}_I := \{B_1^I, \ldots, B_{p+1}^I\}$ is a basis for the space $\mathcal{P}_p$ of polynomials of degree at most p over the interval of interest. We also consider the alternative basis $\mathcal{B}_I := \{\beta_1^I, \ldots, \beta_{p+1}^I\}$, consisting of the B-spline basis functions in $\mathcal{B}$ that are nonzero over I. More precisely, we have that

$$\beta_i^I \equiv \beta_{k+i-p-1}, \qquad \text{on } I, \qquad \forall\, i = 1, \ldots, p+1,$$

where $\operatorname{supp}\beta_{k+i-p-1} = [\xi_{k+i-p-1}, \xi_{k+i}]$. Let $D_I = (d_{ij}) \in \mathbb{R}^{(p+1)\times(p+1)}$ be the change of basis matrix such that

$$B_j^I = \sum_{i=1}^{p+1} d_{ji}\beta_i^I, \qquad \text{for } j = 1, \ldots, p+1. \tag{3}$$

[1] This constant depends on the polynomial degree p, since the number of B-spline basis functions acting on a single mesh element is $p+1$.

For each $j = 1, \dots, p+1$, the coefficients $\{d_{ji}\}_{i=1}^{p+1}$ can be computed by evaluating the blossom of B_j^I via

$$d_{ji} = B_j^I[\xi_{k+i-p}, \dots, \xi_{k+i-1}],$$

see, e.g. [17, page 65]. Thus, we have that (cf. [17, 12])

$$d_{ji} = \frac{1}{(j-1)!(p-j+1)!} \sum_{\sigma\in\Sigma} \left(\prod_{r=1}^{j-1} \frac{\xi_{k+i-\sigma(r)} - \xi_k}{\xi_{k+1} - \xi_k} \prod_{r=j}^{p} \frac{\xi_{k+1} - \xi_{k+i-\sigma(r)}}{\xi_{k+1} - \xi_k} \right),$$

where Σ denotes the set of the permutations in $\{1, \dots, p\}$. In particular, for $j = 1$ we have that

$$\begin{aligned} d_{11} &= \frac{(\xi_{k+1} - \xi_{k-1}) \dots (\xi_{k+1} - \xi_{k+1-p})}{(\xi_{k+1} - \xi_k)^{p-1}} = \frac{\prod_{r=k+1-p}^{k-1} (\xi_{k+1} - \xi_r)}{(\xi_{k+1} - \xi_k)^{p-1}}, && (i = 1) \quad (4) \\ d_{1i} &= 0, && i = 2, \dots, p+1, \end{aligned}$$

and for $j = p+1$,

$$\begin{aligned} d_{p+1,i} &= 0, && i = 1, \dots, p, \\ d_{p+1,p+1} &= \frac{(\xi_{k+2} - \xi_k) \dots (\xi_{k+p} - \xi_k)}{(\xi_{k+1} - \xi_k)^{p-1}} = \frac{\prod_{r=k+2}^{k+p} (\xi_r - \xi_k)}{(\xi_{k+1} - \xi_k)^{p-1}}, && (i = p+1). \end{aligned}$$

Let now consider the case $j = 2, \dots, p$. We may observe that either ξ_k or ξ_{k+1} are in $\{\xi_{k+i-p}, \dots, \xi_{k+i-1}\}$ for i equals to 1 or $p+1$, respectively. Both ξ_k and ξ_{k+1} belong to this knot interval of interest for all the intermediate cases of $i = 2, \dots, p$. At least one of the two can be then fixed and we may consider the remaining nonzero contributions in the sum over the permutations defining d_{ji}, obtaining

$$\begin{aligned} |d_{ji}| &\le \frac{(p-1)!}{(j-1)!(p-j+1)!} \frac{(\xi_{k+i-1} - \xi_{k+i-p})^{p-1}}{(\xi_{k+1} - \xi_k)^{p-1}} \\ &= \frac{1}{p} \begin{pmatrix} p \\ j-1 \end{pmatrix} \frac{(\xi_{k+i-1} - \xi_{k+i-p})^{p-1}}{(\xi_{k+1} - \xi_k)^{p-1}}, \qquad i = 1, \dots, p+1. \end{aligned}$$

By taking into account that $\frac{1}{p} \sum_{j=2}^{p} \begin{pmatrix} p \\ j-1 \end{pmatrix} = \frac{1}{p}(2^p - 2)$, we then obtain

$$\sum_{j=1}^{p+1} |d_{ji}| \le \left(\frac{1}{p}(2^p - 2) + 1 \right) \frac{|\hat{I}|^{p-1}}{|I|^{p-1}}, \qquad i = 1, \dots, p+1.$$

Thus, we have proved the following result.

Lemma 4.1. *Let $I \in \mathcal{I}$ and $D_I = (d_{ij}) \in \mathbb{R}^{(p+1)\times(p+1)}$ be the change of basis matrix satisfying (3). Then,*

$$\|D_I^T\|_\infty = \max_{i=1,\dots,p+1} \sum_{j=1}^{p+1} |d_{ji}| \le c_p \frac{|\hat{I}|^{p-1}}{|I|^{p-1}}, \tag{5}$$

where $c_p := \frac{1}{p}(2^p - 2) + 1$ and $\hat{I}$ is given by (2).

Whereas Assumption 1 allows to bound by above uniformly the right hand side of (5), in the next example we show that this is not the case when only Assumption 2 holds.

Example 4.2. Let $p \geq 2$ and let $\varepsilon > 0$. We consider

$$\Xi := \{\underbrace{0, \dots, 0}_{p+1 \text{ times}}, 1/2, 1/2+\varepsilon, \dots, 1/2+p\varepsilon, \underbrace{1, \dots, 1}_{p+1 \text{ times}}\}.$$

Note that in this case Assumption 2 holds. In particular, we show that it is not possible to bound $\|D_I^T\|_\infty$ uniformly by above, and that, in fact, the behaviour predicted by the right hand side of estimation (5) can be reached. Let $I := [1/2, 1/2+\varepsilon] = [\xi_{p+2}, \xi_{p+3}]$. We have $\hat{I} = [\xi_3, \xi_{2p+2}] = [0, 1/2+p\varepsilon]$, and, consequently, $\left(\frac{|\hat{I}|}{|I|}\right)^{p-1} = \mathcal{O}(\varepsilon^{1-p})$ as $\varepsilon \to 0$. According to (4), we then obtain

$$\|D_I^T\|_\infty \geq |d_{11}| = \left(\frac{1/2+\varepsilon}{\varepsilon}\right)^{p-1} = \mathcal{O}(\varepsilon^{1-p}), \quad \text{as } \varepsilon \to 0.$$

4.2 Local stability of the B-spline basis

Let $Q \in \mathcal{Q}$ and $\mathbf{p} := (p_1, p_2, \dots, p_d)$. We denote by $\mathcal{P}_\mathbf{p}$ the space of tensor-product polynomials with degree at most p_i in the coordinate direction x_i, for $i = 1, 2, \dots, d$. Let $N := \dim \mathcal{P}_\mathbf{p} = \Pi_{i=1}^d (p_i + 1)$. In this section we analyse the local stability of the B-spline basis. More precisely, we study the existence of a constant $C > 0$ (independent of Q) such that

$$\|x\|_\infty \leq C \left\| \sum_{j=1}^N x_j \beta_j^Q \right\|_{L^\infty(Q)}, \qquad \forall\, x = (x_1, \dots, x_N) \in \mathbb{R}^N, \tag{6}$$

where $\beta_1^Q, \dots, \beta_N^Q$ are the B-spline basis functions in $\mathcal{B}$ restricted to Q.

Remark 4.3 (The inverse of the local Bézier extraction operator)**.** We now generalise the results of Section 4.1 to the tensor-product case. Let $Q = I_1 \times \cdots \times I_d \in \mathcal{Q}$ be given. We consider the set $\mathbb{B}_Q := \{B_1^Q, \dots, B_N^Q\}$ of tensor-product Bernstein polynomials on Q, which constitutes a basis for $\mathcal{P}_\mathbf{p}$. On the other hand, we consider the alternative basis $\mathcal{B}_Q := \{\beta_1^Q, \dots, \beta_N^Q\}$, consisting of the B-spline basis functions in $\mathcal{B}$ restricted to Q. Let $D_Q = (d_{ij}) \in \mathbb{R}^{N \times N}$ be the matrix such that

$$B_j^Q = \sum_{i=1}^N d_{ji} \beta_i^Q, \qquad \text{for } j = 1, \dots, N.$$

Notice that D_Q is the matrix for the change of bases and satisfies

$$[f]_{\mathcal{B}_Q} = D_Q^T [f]_{\mathbb{B}_Q}, \qquad \forall\, f \in \mathcal{P}_\mathbf{p}, \tag{7}$$

where $[f]_{\mathcal{B}_Q}$ and $[f]_{\mathbb{B}_Q}$ denote the vector of coefficients for writing f as a linear combination of the functions in $\mathcal{B}_Q$ and $\mathbb{B}_Q$, respectively.

It is easy to check that $D_Q = D_{I_d} \otimes \cdots \otimes D_{I_1}$ and that

$$\|D_Q^T\|_\infty = \prod_{i=1}^d \|D_{I_i}^T\|_\infty,$$

where $D_{I_i}^T$ denotes the corresponding univariate local Bézier extraction operator defined in Section 4.1, for $i = 1, \dots, d$. In view of (5) we have that

$$\|D_Q^T\|_\infty \leq C_{\mathbf{p}} \prod_{i=1}^{d} \frac{|\hat{I}_i|^{p_i-1}}{|I_i|^{p_i-1}}, \tag{8}$$

where $C_{\mathbf{p}} := \prod_{i=1}^d c_{p_i} = \prod_{i=1}^d (\frac{1}{p_i}(2^{p_i} - 2) + 1)$. We remark that (8) generalises Lemma 4.1 for the tensor-product case. Notice that under Assumption 1 (in each coordinate direction), we can bound $\|D_Q^T\|_\infty$ uniformly by a constant which depends only on $\mathbf{p}$ and θ.

Remark 4.4 (L^∞-local stability of the Bernstein basis)**.** Let $Q = [0, 1]^d$. Using the fact that all the norms are equivalent in $\mathcal{P}_{\mathbf{p}}$ we have that there exists a constant $C_{SB} > 0$ depending only on $\mathbf{p}$ such that

$$\|x\|_\infty \leq C_{SB} \left\| \sum_{j=1}^{N} x_j B_j^Q \right\|_{L^\infty(Q)}, \tag{9}$$

for all $x = (x_1, \dots, x_N) \in \mathbb{R}^N$. The same result holds for an arbitrary rectangle $Q \subset \mathbb{R}^d$.

Now, using (7) and (9) we have the following result.

Lemma 4.5 (L^∞-local stability of the B-spline basis)**.** *Let $Q \in \mathcal{Q}$. Then,*

$$\|x\|_\infty \leq C_{SB} \|D_Q^T\|_\infty \left\| \sum_{j=1}^{N} x_j \beta_j^Q \right\|_{L^\infty(Q)},$$

for all $x = (x_1, \dots, x_N) \in \mathbb{R}^N$.

Under Assumption 1, taking into account (8), we can bound $\|D_Q^T\|_\infty$ uniformly by a constant which depends only on $\mathbf{p}$ and θ. In this case, we have that the B-spline basis is L^∞-locally stable (see also [12]). On the other hand, under Assumption 2, estimate (6) does not hold, as it is showed in the following example.

Example 4.6. If we consider again the open-knot vector Ξ of Example 4.2, we have that

$$B_1^I = d_{11} \beta_1^I,$$

where $d_{11} = \left(\frac{1/2+\varepsilon}{\varepsilon}\right)^{p-1}$. Since $\|B_1^I\|_{L^\infty(I)} = 1$ we obtain

$$\left(\frac{1/2 + \varepsilon}{\varepsilon}\right)^{p-1} = \frac{d_{11}}{1} \leq \sup_{x \in \mathbb{R}^{p+1}} \frac{\|x\|_\infty}{\left\| \sum_{j=1}^{p+1} x_j \beta_j^I \right\|_{L^\infty(I)}} \longrightarrow \infty,$$

as $\varepsilon \to 0$.

5 Local approximation methods

Regarding the stability estimation of Lemma 4.5, in this section we present a local approximation method.

Lemma 5.1 (Local L^2-projection). *Let $Q \in \mathcal{Q}$ and let $\Pi_Q : L^1(Q) \to \mathcal{P}_{\mathbf{p}}$ be the L^2-projection operator defined by*

$$\int_Q (f - \Pi_Q f) g = 0, \qquad \forall\, g \in \mathcal{P}_{\mathbf{p}}. \tag{10}$$

Then, there exists a constant $C_\Pi > 0$ depending only on $\mathbf{p}$ such that

$$\|\Pi_Q f\|_{L^\infty(Q)} \leq C_\Pi |Q|^{-1} \|f\|_{L^1(Q)}, \qquad \forall\, f \in L^1(Q).$$

Proof. Let $f \in L^1(Q)$. From the definition of Π_Q it follows that

$$\|\Pi_Q f\|^2_{L^2(Q)} = \int_Q f \Pi_Q f \leq \|f\|_{L^1(Q)} \|\Pi_Q f\|_{L^\infty(Q)}.$$

On the other hand, since $\mathcal{P}_{\mathbf{p}}$ is a finite dimensional space, we have that there exists a constant $C_I > 0$ depending only on $\mathbf{p}$ such that the following inverse inequality holds:

$$\|g\|_{L^\infty(Q)} \leq C_I |Q|^{-\frac{1}{2}} \|g\|_{L^2(Q)}, \qquad \forall\, g \in \mathcal{P}_{\mathbf{p}}.$$

Therefore,

$$\|\Pi_Q f\|^2_{L^\infty(Q)} \leq C_I^2 |Q|^{-1} \|\Pi_Q f\|^2_{L^2(Q)} \leq C_I^2 |Q|^{-1} \|f\|_{L^1(Q)} \|\Pi_Q f\|_{L^\infty(Q)},$$

and thus,

$$\|\Pi_Q f\|_{L^\infty(Q)} \leq C_I^2 |Q|^{-1} \|f\|_{L^1(Q)}.$$

□

Remark 5.2. Notice that

$$\Pi_Q f = \sum_{i=1}^{N} \lambda_i^Q(f) \beta_i^Q, \qquad \forall\, f \in L^1(Q), \tag{11}$$

where $\lambda^Q(f) := (\lambda_1^Q(f), \ldots, \lambda_N^Q(f))^T$ is the solution of the linear system

$$M_Q \mathbf{x} = F_Q,$$

where

$$M_Q = \left(\int_Q \beta_j^Q \beta_i^Q \right)_{i,j=1,\ldots,N} \in \mathbb{R}^{N \times N}, \quad \text{and} \quad F_Q = \left(\int_Q f \beta_i^Q \right)_{i=1,\ldots,N} \in \mathbb{R}^{N \times 1}.$$

On the other hand, $\{\lambda_i^Q : L^1(Q) \to \mathbb{R} \,|\, i = 1, \ldots, N\}$ is a dual basis for $\mathcal{B}_Q$ in the sense that

$$\lambda_i^Q(\beta_j^Q) = \delta_{ij}, \qquad i, j = 1, \ldots, N. \tag{12}$$

As a consequence of the L^∞-local stability of the B-spline basis (Lemma 4.5) we can state the following result.

Theorem 5.3. *Let $Q \in \mathcal{Q}$ and let $\Pi_Q : L^1(Q) \to \mathcal{P}_{\mathbf{p}}$ be the L^2-projection operator defined by (10). Let q be such that $1 \le q \le \infty$. Then,*

$$\|\lambda^Q(f)\|_\infty \le C_\Pi C_{SB} \|D_Q^T\|_\infty |Q|^{-\frac{1}{q}} \|f\|_{L^q(Q)}, \qquad \forall\, f \in L^q(Q), \tag{13}$$

where $\lambda^Q(f) = (\lambda_1^Q(f), \ldots, \lambda_N^Q(f))^T$ are the coefficients of $\Pi_Q(f)$ with respect to the local B-spline basis $\mathcal{B}_Q$ (cf. (11)).

Proof. Let q be such that $1 \le q \le \infty$ and $f \in L^q(Q)$. Using Lemmas 4.5 and 5.1 we have that

$$\|\lambda^Q(f)\|_\infty \le C_{SB} \|D_Q^T\|_\infty \, \|\Pi_Q f\|_{L^\infty(Q)} \le C_\Pi C_{SB} \|D_Q^T\|_\infty |Q|^{-1} \|f\|_{L^1(Q)}.$$

Finally, (13) is as consequence of Hölder inequality. □

6 Locally supported dual basis for B-splines

The goal of this section is to define a dual basis for the multivariate B-spline basis $\mathcal{B}$, i.e., a set of linear functionals

$$\{\lambda_\beta : L^1(\Omega) \to \mathbb{R} \,|\, \beta \in \mathcal{B}\},$$

such that $\lambda_{\beta_i}(\beta_j) = \delta_{ij}$, for all $\beta_i, \beta_j \in \mathcal{B}$. More precisely, we are interested in defining such functionals satisfying the following properties:

(i) *Local support:* λ_β is *supported* in $\Lambda_\beta \subset \operatorname{supp}\beta$, i.e.,

$$\forall\, f \in L^1(\Omega), \quad f_{|\Lambda_\beta} \equiv 0 \qquad \Longrightarrow \qquad \lambda_\beta(f) = 0.$$

(ii) *Dual basis:* For $\beta_i, \beta_j \in \mathcal{B}$, $\lambda_{\beta_i}(\beta_j) = \delta_{ij}$.

(iii) *L^q-Stability:* Let $1 \le q \le \infty$. There exists a constant $C_S > 0$ such that

$$|\lambda_\beta(f)| \le C_S |\operatorname{supp}\beta|^{-\frac{1}{q}} \|f\|_{L^q(\operatorname{supp}\beta)}, \qquad \forall\, f \in L^q(\Omega),\ \beta \in \mathcal{B}. \tag{14}$$

Remark 6.1. Condition (iii) will be a key tool for proving the local stability of a quasi-interpolant operator in Section 7.

We will use the technique in [15] to define linear functionals $\{\lambda_\beta\}_{\beta\in\mathcal{B}}$ satisfying the desired properties. Roughly speaking, we define the functional λ_β as a convex combination of the local projections onto some $Q \in \mathcal{Q}$ such that $Q \subset \operatorname{supp}\beta$. For $\beta \in \mathcal{B}$, we define

$$\mathcal{Q}_\beta := \{Q \in \mathcal{Q} \,|\, Q \subset \operatorname{supp}\beta\},$$

and for each $Q \in \mathcal{Q}_\beta$, let $\lambda_\beta^Q := \lambda_{i_0}^Q$, where $i_0 = i_0(\beta, Q)$ with $1 \le i_0 \le N$ is such that $\beta_{i_0}^Q \equiv \beta$ on Q. Thus, the functional λ_β is given by

$$\lambda_\beta := \sum_{Q \in \mathcal{Q}_\beta} c_{Q,\beta} \lambda_\beta^Q, \tag{15}$$

where

$$\forall\, Q \in \mathcal{Q}_\beta,\ c_{Q,\beta} \geq 0, \qquad \text{and} \qquad \sum_{Q\in\mathcal{Q}_\beta} c_{Q,\beta} = 1. \tag{16}$$

Notice that λ_β is supported in $\Lambda_\beta := \bigcup_{\substack{Q\in\mathcal{Q}_\beta \\ c_{Q,\beta}>0}} Q \subset \operatorname{supp}\beta$, and therefore, condition (i) holds. On the other hand, condition (ii) is a consequence of (12) and (16). In the rest of this section we analyse the validity of (iii).

We propose some possible choices for the coefficients $c_{Q,\beta}$, in order to guarantee the validity of condition (iii) under different assumptions on the underlying univariate meshes.

Case 1. Locally quasi-uniform meshes (Assumption 1) Under the Assumption 1 there is no restriction on the choice of the coefficients $c_{Q,\beta}$, i.e., condition (iii) always holds. In fact, taking into account (13), we have that (14) holds with a constant $C_S > 0$ which depends on $\mathbf{p}$ and θ.

Case 2. Non locally quasi-uniform meshes (Assumption 2) If Assumption 1 does not hold, but Assumption 2 does, we propose two ways of defining $c_{Q,\beta}$ in order to obtain the validity of condition (iii).

(1) Let[2] $c_\beta := \displaystyle\sum_{Q\in\mathcal{Q}_\beta} \prod_{i=1}^{d} \left(\frac{|I_i|}{|\tilde{I}_i|}\right)^{p_i-1+\frac{1}{q}}$ and

$$c_{Q,\beta} := \frac{1}{c_\beta} \prod_{i=1}^{d} \left(\frac{|I_i|}{|\tilde{I}_i|}\right)^{p_i-1+\frac{1}{q}}, \qquad \forall\, Q \in \mathcal{Q}_\beta.$$

Then, using the definition of λ_β given by (15) and the bound for λ_β^Q given by Theorem 5.3 we have that

$$|\lambda_\beta(f)| \leq C_\Pi C_{SB} \sum_{Q\in\mathcal{Q}_\beta} c_{Q,\beta} \|D_Q^T\|_\infty |Q|^{-\frac{1}{q}} \|f\|_{L^q(Q)}.$$

Now, regarding the definition of $c_{Q,\beta}$ and the bound for $\|D_Q^T\|_\infty$ given in (8), we obtain

$$|\lambda_\beta(f)| \leq \frac{C_\Pi C_{SB} C_{\mathbf{p}}}{c_\beta} \sum_{Q\in\mathcal{Q}_\beta} |\tilde{Q}|^{-\frac{1}{q}} \|f\|_{L^q(Q)}.$$

Since $|\operatorname{supp}\beta| \leq |\tilde{Q}|$, for all $Q \in \mathcal{Q}_\beta$,

$$|\lambda_\beta(f)| \leq \frac{C_\Pi C_{SB} C_{\mathbf{p}} (\#\mathcal{Q}_\beta)^{1-\frac{1}{q}}}{c_\beta} |\operatorname{supp}\beta|^{-\frac{1}{q}} \|f\|_{L^q(\operatorname{supp}\beta)}.$$

[2] For each Q we consider the representation $Q = I_1 \times \cdots \times I_d$.

Under the Assumption 2,[3] we can bound c_β uniformly by below. More precisely, if $p_{\max} := \max_{i=1,\dots,d} p_i$, then

$$\begin{aligned} c_\beta &= \sum_{Q\in\mathcal{Q}_\beta} \prod_{i=1}^{d} \left(\frac{|I_i|}{|\tilde{I}_i|}\right)^{p_i-1+\frac{1}{q}} \geq \sum_{Q\in\mathcal{Q}_\beta} \left(\prod_{i=1}^{d} \frac{|I_i|}{|\tilde{I}_i|}\right)^{p_{\max}-1+\frac{1}{q}} \\ &= \sum_{Q\in\mathcal{Q}_\beta} \left(\frac{|Q|}{|\tilde{Q}|}\right)^{p_{\max}-1+\frac{1}{q}} \geq (\#\mathcal{Q}_\beta)^{2-p_{\max}-\frac{1}{q}} \left(\sum_{Q\in\mathcal{Q}_\beta} \frac{|Q|}{|\tilde{Q}|}\right)^{p_{\max}-1+\frac{1}{q}} \\ &\geq (\#\mathcal{Q}_\beta)^{2-p_{\max}-\frac{1}{q}} \frac{1}{C_2^{d(p_{\max}-1+\frac{1}{q})}} \left(\underbrace{\sum_{Q\in\mathcal{Q}_\beta} \frac{|Q|}{|\operatorname{supp}\beta|}}_{=1}\right)^{p_{\max}-1+\frac{1}{q}}. \end{aligned}$$

Thus, $\frac{1}{c_\beta} \leq \frac{C_2^{d(p_{\max}-1+\frac{1}{q})}}{(\#\mathcal{Q}_\beta)^{2-p_{\max}-\frac{1}{q}}}$, and

$$|\lambda_\beta(f)| \leq C_\Pi C_{SB} C_{\mathbf{p}} (\#\mathcal{Q}_\beta)^{p_{\max}-1} C_2^{d(p_{\max}-1+\frac{1}{q})} |\operatorname{supp}\beta|^{-\frac{1}{q}} \|f\|_{L^q(\operatorname{supp}\beta)},$$

which in turn implies (14).

(2) For each β we associate an element $Q_\beta \in \mathcal{Q}_\beta$ with size equivalent to the size of its support, i.e., such that

$$\frac{|\operatorname{supp}\beta|}{|Q_\beta|} \leq C,$$

with a constant C depending on $\mathbf{p}$. For example, we can select $Q_\beta \in \arg\max_{Q\in\mathcal{Q}_\beta} |Q|$ and in this case $\frac{|\operatorname{supp}\beta|}{|Q_\beta|} \leq \#\mathcal{Q}_\beta \leq N$.

Under the Assumption 2 we have that

$$\|D^T_{Q_\beta}\|_\infty \leq C_{\mathbf{p}} \left(\frac{|\tilde{Q}_\beta|}{|Q_\beta|}\right)^{p_{\max}-1} \leq C_{\mathbf{p}} (C_2^d C)^{p_{\max}-1}.$$

In this case, we define $c_{Q,\beta} := \begin{cases} 1, & \text{if } Q = Q_\beta \\ 0, & \text{if } Q \neq Q_\beta \end{cases}$, i.e., $\lambda_\beta = \lambda_\beta^{Q_\beta}$. Finally, (14) follows from Theorem 5.3.

7 Quasi-interpolation in spline spaces

We consider a dual basis $\{\lambda_\beta\}_{\beta\in\mathcal{B}}$ from (15), satisfying conditions (i)-(ii)-(iii) stated in the previous section, for some q such that $1 \leq q \leq \infty$. Let $P : L^q(\Omega) \to \mathbb{V} = \operatorname{span}\mathcal{B}$ be

[3] Without loss of generality, we denote by C_2 the constant in Remark 3.1 for each coordinate direction, and thus, $\frac{|\tilde{Q}|}{|\operatorname{supp}\beta|} \leq C_2^d$, for all $Q \in \mathcal{Q}$ such that $Q \subset \operatorname{supp}\beta$.

given by

$$P(f) := \sum_{\beta \in \mathcal{B}} \lambda_\beta(f)\beta, \qquad \forall\, f \in L^q(\Omega). \tag{17}$$

The next result states some important properties of P.

Theorem 7.1. *The following holds:*

(a) P is a projection on $\mathbb{V}$, i.e., for all $f \in \mathbb{V}$, $P(f) = f$.

(b) Local stability*: Let $1 \le q \le \infty$. For $Q = I_1 \times \ldots \times I_d \in \mathcal{Q}$, the operator P satisfies*

$$\|Pf\|_{L^q(Q)} \le C_S \|f\|_{L^q(\tilde{Q})}, \qquad \forall\, f \in L^q(\Omega).$$

where $\tilde{Q} = \tilde{I}_1 \times \ldots \times \tilde{I}_d$ denotes the support extension (see (1)) and $C_S > 0$ is the constant appearing in (14).

(c) Local approximation*: Let $\mathbf{s} := (s_1, \ldots, s_d)$ be such that $0 \le s_i \le p_i + 1$, for $i = 1, \ldots, d$. Then, there exists a constant $C_A > 0$ such that, for $Q = I_1 \times \ldots \times I_d \in \mathcal{Q}$, it holds that*

$$\|f - Pf\|_{L^q(Q)} \le C_A \sum_{i=1}^{d} |\tilde{I}_i|^{s_i} \|D^{s_i}_{x_i} f\|_{L^q(\tilde{Q})}, \qquad \forall\, f \in W^{q,\mathbf{s}}(\Omega),$$

where $W^{q,\mathbf{s}}(\Omega) := \{f \in L^q(\Omega) \ :\ D^{r_i}_{x_i} f \in L^q(\Omega),\ 0 \le r_i \le s_i,\ i = 1, \ldots, d\}$.

Proof. (a) It is an immediate consequence of condition (ii).

(b) Let $Q \in \mathcal{Q}$. Then, taking into account the definition of P given by (17), the spline partition-of-unity property and the L^q-stability in (14), we have

$$|P(f)| \le \max_{\substack{\beta \in \mathcal{B} \\ \operatorname{supp} \beta \supset Q}} |\lambda_\beta(f)| \le C_S |Q|^{-\frac{1}{q}} \|f\|_{L^q(\tilde{Q})}, \qquad \text{on } Q.$$

Therefore,

$$\|P(f)\|_{L^q(Q)} \le C_S \|f\|_{L^q(\tilde{Q})}.$$

(c) Let $Q \in \mathcal{Q}$. By the classical polynomial approximation property, there exists $p_{\tilde{Q}} \in \mathcal{P}_{\mathbf{p}}$ such that

$$\|f - p_{\tilde{Q}}\|_{L^q(\tilde{Q})} \le C_T \sum_{i=1}^{d} |\tilde{I}_i|^{s_i} \|D^{s_i}_{x_i} f\|_{L^q(\tilde{Q})}, \tag{18}$$

where the constant $C_T > 0$ only depends on d, $\mathbf{p}$, $\mathbf{s}$ and q. Taking into account the local stability of P given in (b) and (18) we have that

$$\begin{aligned}
\|f - Pf\|_{L^q(Q)} &\le \|f - p_{\tilde{Q}}\|_{L^q(Q)} + \|p_{\tilde{Q}} - Pf\|_{L^q(Q)} \\
&= \|f - p_{\tilde{Q}}\|_{L^q(Q)} + \|P(p_{\tilde{Q}} - f)\|_{L^q(Q)} \\
&\le (1 + C_S)\|f - p_{\tilde{Q}}\|_{L^q(\tilde{Q})} \\
&\le (1 + C_S) C_T \sum_{i=1}^{d} |\tilde{I}_i|^{s_i} \|D^{s_i}_{x_i} f\|_{L^q(\tilde{Q})}.
\end{aligned}$$

□

Remark 7.2. The operator defined in [21] and called Bézier projection fits in this framework and consists in a specific choice of coefficients $c_{Q,\beta}$ in (15). Our results of Section 6 provide stability for this operator under Assumption 1.

8 Conclusions

We have defined a class of quasi-interpolation operators onto spline spaces that enjoy L^2 stability properties and optimal locality properties, under very general assumption on the knot distributions. These operators are proved to deliver optimal approximation properties with respect to h for tensor product spline spaces. It should be noted though that the behaviour of constants with respect to the degree p is not analysed and is likely not optimal.

The class of operators we consider are associated with the construction of a dual basis and for this reason, they can be used in situations that are more general than tensor product B-splines. In particular, following [5] and [4], it is clear that the same construction would provide a dual basis in the case of analysis suitable (or dual compatible) T-splines (see [3] and the references there in). In the same lines, following [20] and [7], our class of operators can be used to provide quasi-interpolation operators for hierarchical splines (see [14], [22]) as well. The analysis presented in this paper can provide a general framework for the study of the local approximation properties for such quasi-interpolants on either T-splines or Hierarchical splines, but such results are beyond the scope of the present contribution.

Acknowledgements

Annalisa Buffa and Giancarlo Sangalli were partially supported by the European Research Council through the FP7 ERC Consolidator Grant n.616563 *HIGEOM*, and by the Italian MIUR through the PRIN "Metodologie innovative nella modellistica differenziale numerica". Eduardo M. Garau was partially supported by CONICET through grant PIP 112-2011-0100742, by Universidad Nacional del Litoral through grants CAI+D 500 201101 00029 LI, 501 201101 00476 LI, by Agencia Nacional de Promoción Científica y Tecnológica, through grants PICT-2012-2590 and PICT-2014-2522 (Argentina). Carlotta Giannelli was supported by the project DREAMS (MIUR "Futuro in Ricerca" RBFR13FBI3) and by the Gruppo Nazionale per il Calcolo Scientifico (GNCS) of the Istituto Nazionale di Alta Matematica (INdAM). This support is gratefully acknowledged.

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