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Keywords: *Isogeometric analysis, B-splines, NURBS, Collocation method, Superconvergent points*

Optimal-order isogeometric collocation at Galerkin superconvergent points

M. Montardini[*], G. Sangalli[†], L. Tamellini[‡]

September 7, 2016

Abstract In this paper we investigate numerically the order of convergence of an isogeometric collocation method that builds upon the least-squares collocation method presented in [1] and the variational collocation method presented in [14]. The focus here is on smoothest B-splines/NURBS approximations, i.e, having global C^{p-1} continuity for polynomial degree p. In particular, we show that using as collocation points a suitable subset of those considered in [1] (which are related to the Galerkin superconvergence theory) it is possible to achieve optimal L^2-convergence for odd degree B-splines/NURBS approximations with a pure collocation scheme, i.e., considering as many collocation points as degrees-of-freedom. The method in [1], instead, is based on a least-squares formulation due to the fact that the set of collocation points outnumbers the degrees-of-freedom to be computed. We especially highlight that we obtain fourth-order convergence for the L^2 and L^∞ norm of the error when considering cubic basis functions. Further careful analysis is however needed, since the robustness of the method and its mathematical foundations are still unclear.

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1 Introduction

The splines-based collocation method for solving differential equations has about fifty years of history. The first references are [7, 13], where cubic C^2 splines are used to solve a second order two-point boundary value problem. In particular, in order to achieve optimal convergence, [13] collocates a modified equation, where the modification is obtained by constructing a suitable interpolant of the true solution. An extension of this approach to multivariate (tensor-product) splines and partial differential equations is studied in [16], while extensions to m-order differential equations are found in [24] and in particular in [11], where the optimality of the method is achieved by collocating the original, unperturbed, equation at suitably selected collocation points, that is, Gaussian quadrature points. The method only works for splines of continuity C^{m-1} and degree

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$m + k - 1$, with $k \geq m$. Splines-based collocation has been successfully applied also to integro-differential equations on curves, and to the boundary element method for planar domains (see [2] and references therein).

The interest and development of splines-based collocation methods for partial differential equations has been driven in the last decade by isogeometric analysis (see [5, 4, 3, 6, 17, 22, 12, 23, 9, 19, 15, 18, 14] and references therein). The motivation is computational efficiency: isogeometric collocation is more efficient than the isogeometric Galerkin method, at least for standard code implementations, see [25]. In particular, the assembly of system matrices is much faster for collocation than for Galerkin (unless one adopts recent quadrature algorithms as in [8]). On the other hand, contrary to the Galerkin method, isogeometric collocation based on maximal regularity splines has always been reported suboptimal in literature, when the error is measured in L^2 or L^∞ norm. For example, the L^2 norm of the error of the collocation method at Greville points, studied in [5] for a second-order elliptic problem, converges under h-refinement as $O(h^{p-1})$ or $O(h^p)$, when the degree p is odd or even, respectively, while the optimal interpolation error is $O(h^{p+1})$ regardless of the parity of p for a smooth solution. We remark that the previous ideas of [13, 11] cannot be applied directly to the isogeometric case since [13] would require a complex modification of the equation (this approach however deserves further investigation) and [11] does not work for maximal smoothness splines, which represent the most interesting choice in this framework.

Collocating the equation at Greville points (obtaining the method to which we refer here as Collocation at Greville Points, C-GP), is a common choice since Greville points are classical interpolation points for arbitrary degree and regularity splines, well studied in literature, see e.g. [10]. There is however an interesting alternative, inspired by [11] and stated in [1], which seeks for collocation points such that the collocation solution coincides with the Galerkin one, thus recovering optimal convergence. These points are named Cauchy-Galerkin points, and are the zeros of the Galerkin residual (see [14] for details). Unfortunately, these points are unavailable a-priori; therefore, [1] selects as a surrogate the points where, under suitable hypotheses, superconvergence of the second derivatives of the Galerkin solution occurs, motivated by the fact that for a Poisson problem the Galerkin residual is actually equivalent to the error on the approximation of the second derivatives (we will return on this point later on). Thus, if the collocation method constrains the residual to be zero where the Galerkin residual is estimated to be zero at least up to higher order terms, there is hope for the collocation solution to be close to the Galerkin solution. However, since there are more Galerkin superconvergent points than degrees-of-freedom, n_{dof}, for maximal smoothness splines (the superconvergent points are about $2n_{dof}$), [1] proposes to compute a solution of the overdetermined linear system by a least-square approximation. This approach, which is not collocation in a strict sense, gives optimal convergence for odd degrees and one-order suboptimal for even degrees. We refer to it as Least-Squares approximation at Superconvergent Points (LS-SP).

The idea above is further developed in [14], where a standard collocation formulation is obtained by selecting only n_{dof} collocation points among those used in [1], and showing that a well-posed collocation scheme can be obtained if, roughly speaking, one superconvergent point per element is used as collocation point; since this implies using every other superconvergent point (as shall be clearer later), we refer to this method as Collocation at Alternating Superconvergent Points (in short C-ASP). The L^2 convergence of C-ASP is one-order suboptimal for *any* degree p, i.e., the L^2-error decays as $O(h^p)$ for any p, and can therefore be seen in some sense as an "intermediate" method between C-GP and LS-SP. In other words, the C-ASP solution is not as close as expected to the Galerkin solution. The reason is that the superconvergence points can be computed only when assumptions are made, e.g., on the local symmetry of the mesh (see [26]) or on the periodicity of the error (see [1], [14]) and these assumptions does not hold true everywhere in our computational domain. Nonetheless, the only practical possibility is to work with the superconvergent points derived under the above-mentioned assumptions also when such assumptions do not hold true. In this sense, it is not correct to refer to these points as "superconvergent points", but we do so in this paper for the sake of exposition. The correct and complete identification of superconvergent points for problems of interest remains an open challenge.

Our present work continues in the direction of [14]. We propose a new criterion for selecting a subset of superconvergent points to obtain a standard collocation method, yet with better convergence properties than the C-ASP. This can be achieved by roughly taking two superconvergent points in every other element. This method, which we name Clustered Superconvergent Points (C-CSP), features the same convergence order as the LS-SP approach, i.e., optimal convergence for odd degrees in L^2 and L^∞ norm. Thus, we finally achieve optimally convergent isogeometric collocation with cubic C^2 splines.

The results we have obtained are preliminary and, while something "magic" happens with the C-CSP collocation point selection, we are still unable to provide a rigorous convergence proof for C-CSP (nor for LS-SP or C-ASP). Furthermore, we have considered quite simple numerical benchmarks, therefore the numerical evidence that we have gathered is not yet conclusive regarding the robustness of the method. C-CSP definitely deserves further analysis.

The outline of this work is as follows. Section 2 is a quick overview on B-splines, NURBS, and isogeometric analysis. In Section 3 we present a framework for isogeometric collocation and the collocation schemes C-GP, LS-SP, C-ASP, and the new C-CSP. In Section 4 we show some numerical tests of C-CSP, focusing on the odd degree case, discuss its robustness and compare it with the other collocation methods. Finally, some conclusions and perspective on future works are detailed in Section 5.

2 Preliminaries

2.1 B-splines

Let us consider an interval $\hat{\Omega} \subset \mathbb{R}$. The B-splines basis functions defined on $\hat{\Omega}$ are piecewise polynomials that are built from a *knot vector*, i.e. a vector with non-decreasing entries $\Xi = [\xi_1, \xi_2 ..., \xi_{n+p+1}]$, where n and p are, respectively, the number of basis function that will be built from the knot vector and their polynomial degree. We name *element* a *knot span* (ξ_i, ξ_{i+1}) having non-zero length, and we denote by h the maximal length (the *meshsize*). A knot vector is said to be *open* if its first and last knot have multiplicity $p + 1$, i.e., each of them is repeated $p + 1$ times.

Following [10] and given a knot vector Ξ, univariate B-splines basis functions $N_{i,p}$ are defined recursively as follows for $i = 1, \dots, n$:

$$N_{i,0}(\xi) = \begin{cases} 1 & \xi_i \le \xi < \xi_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

$$N_{i,p}(\xi) = \begin{cases} \dfrac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \dfrac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi), & \xi_i \le \xi < \xi_{i+p+1} \\ 0, & \text{otherwise} \end{cases}$$

where we adopt the convention $\dfrac{0}{0} = 0$; note that the basis corresponding to an open knot vector will be interpolatory in the first and last knot.

Remark 2.1. *In this work we only consider knot vectors whose internal knots have multiplicity one: the associated B-splines/NURBS have then global C^{p-1} regularity.*

We define by $\hat{S}^p = \text{span}\{N_{i,p} | i = 1, ..., n\}$ the space spanned by B-splines of degree p and regularity $p - 1$, built from a given knot vector Ξ. We also introduce the space of periodic B-splines, spanning the space $\widetilde{S}^p = \{v \in \hat{S}^p | v(0) = v(1),\ v'(0) = v'(1), ...,\ v^{(p-1)}(0) = v^{(p-1)}(1)\}$; interestingly, the dimension of $\widetilde{S}^p$ equals the number of elements of the underlying knot vector Ξ, a property that will come in handy later on.

Multivariate splines spaces can constructed from univariate spaces by means of tensor products. For example, a B-splines space in two dimensions can be defined by considering the knot vectors $\Xi = [\xi_1, \xi_2 ..., \xi_{n+p+1}]$ and $\Lambda = [\eta_1, \eta_2, ..., \eta_{m+q+1}]$, and defining $\hat{S}^{p,q} = \text{span}\{N_{i,p}(\xi) M_{j,q}(\eta), i = 1, ..., n, j = 1, ..., m\}$. In the following, it will be useful to refer to the basis functions spanning $\hat{S}^{p,q}$ with a single running index k ranging from 1 to $n \times m$, i.e.

$$\hat{S}^{p,q} = \text{span}\{\varphi_k^{p,q}(\xi, \eta)\ | k = i + (j-1)m,\ i = 1, \dots, n,\ j = 1, \dots, m\}. \tag{2.2}$$

2.2 NURBS

Non-uniform rational B-splines (NURBS, cf. [21]) are defined for the purpose of describing geometries of practical interest like conic sections, see e.g. Problem 3 in next section. The definition of a generic bivariate NURBS function on the parametric square $\hat{\Omega}$ is

$$\forall (\xi, \eta) \in \hat{\Omega}, \quad R_{i,j}^{p,q}(\xi, \eta) = \frac{N_{i,p}(\xi) M_{j,q}(\eta) w_{i,j}}{\sum_{\hat{i}=1}^{n} \sum_{\hat{j}=1}^{m} N_{\hat{i},p}(\xi) M_{\hat{j},q}(\eta) w_{\hat{i},\hat{j}}} \forall i = 1, ..., n, \forall j = 1, ..., m$$

where w_i are suitable weights, and $N_{i,p}(\xi), M_{j,q}(\eta)$ are the univariate B-splines basis functions defined in (2.1). Similarly to (2.2) we also introduce a single running index $k = 1, \dots, n \times m$ to refer to the NURBS basis, i.e.,

$$R_k^{p,q}(\xi, \eta) = R_{i,j}^{p,q}(\xi, \eta), \text{ with } k = i + (j-1)m,\ i = 1, \dots, n,\ j = 1, \dots, m.$$

3 Isogeometric collocation and the choice of the collocation points

3.1 Isogeometric collocation

In our numerical tests we will consider both one-dimensional and two-dimensional elliptic problems, which we now introduce.

Problem 1 (One-dimensional Dirichlet boundary problem)**.** *Find* $u : [0,1] \to \mathbb{R}$ *such that*

$$\begin{cases} u''(x) + a_1(x)u'(x) + a_0(x)u(x) = f(x) & \forall x \in (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{3.1}$$

where $a_0, a_1, f : [0,1] \to \mathbb{R}$ *are sufficiently regular functions.*

We assume that this problem has a unique smooth solution. We then look for an approximate solution $u_h(x) = \sum_{i=1}^{n} c_i N_{i,p}(x) \in \hat{S}^p$, that complies with the boundary conditions $u(0) = u(1) = 0$ (i.e. $c_1 = c_n = 0$, since B-splines built on open knot vectors, as in this case, are interpolatory at the first and last knot), and that satisfies (3.1) in $n-2$ *collocation points* $\{\tau_1, ..., \tau_{n-2}\}$ that need to be specified, i.e.

$$u_h''(\tau_i) + a_1 u_h'(\tau_i) + a_0 u_h(\tau_i) = f(\tau_i), \qquad \forall i = 1, ..., n-2. \tag{3.2}$$

The coefficients $c_2, \ldots, c_{n-1}$ are then computed by solving the linear system obtained by inserting the expansion $u_h(x) = \sum_{i=1}^{n} c_i N_{i,p}(x)$ into (3.2). We also shall introduce a periodic version of Problem 1, which we consider because it is particularly simple to set up a collocation scheme for it, due to the already-mentioned fact that the number of degrees-of-freedom n of $\widetilde{S}^p$ (hence the number of collocation points to be used) is identical to the number of elements of Ξ.

Problem 2 (One-dimensional periodic boundary problem)**.** *Find* $u : \mathbb{R} \to \mathbb{R}$ *such that*

$$\begin{cases} u''(x) + a_1 u'(x) + a_0 u(x) = f(x) & \forall x \in \mathbb{R}, \\ u(x) = u(1+x) & \forall x \in \mathbb{R}, \end{cases} \tag{3.3}$$

where a_0, a_1 *and* f *are sufficiently regular periodic functions.*

We assume again that this problem has a unique (periodic) smooth solution. Note that the periodic problem is not well-posed if a_0 is null. The B-splines approximation of the solution of (3.3) is therefore $u_h \in \widetilde{S}^p$ such that

$$u_h''(\tau_i) + a_1 u_h'(\tau_i) + a_0 u_h(\tau_i) = f(\tau_i) \qquad \forall i = 1, ..., n, \tag{3.4}$$

for suitably chosen collocation points $\{\tau_1, ..., \tau_n\}$ with periodic distribution on $[0,1]$.

Finally, we also consider the two-dimensional Poisson equation, that we will solve by a multivariate collocation scheme constructed by tensorizing univariate sets of collocation points. More specifically, we denote by $\Omega \subset \mathbb{R}^2$ a domain described by a NURBS parametrization $\mathbf{F} : \hat{\Omega} \to \Omega$, where $\hat{\Omega} = [0,1] \times [0,1]$ and

$$\mathbf{F}(\xi, \eta) = \sum_{k=1}^{n \times m} \mathbf{P}_k R_k^{p,q}(\xi, \eta), \quad \mathbf{P}_k \in \mathbb{R}^2,$$

we let Γ denote the boundary of Ω, and we consider the Dirichlet problem

Problem 3 (Dirichlet boundary problem)**.** *Find* $u : \Omega \to \mathbb{R}$ *such that*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{3.5}$$

where $f : \Omega \to \mathbb{R}$ *is a sufficiently regular function.*

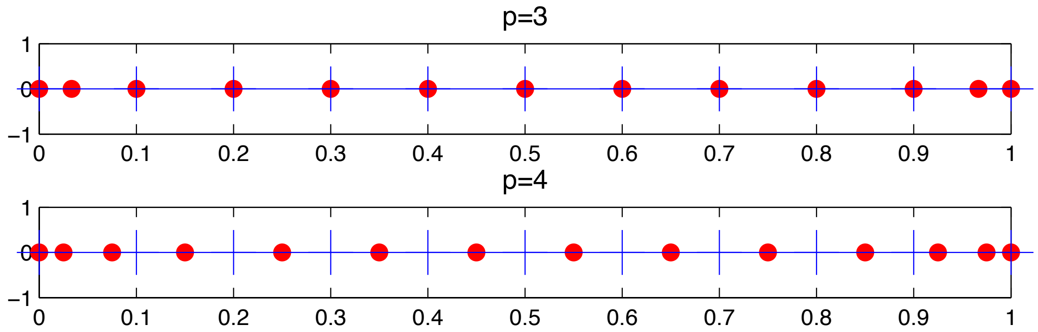


Figure 1: Examples of Greville points computed from an open knot vector: p=3 and p=4. The interior Greville points are used as collocation points in the C-GP scheme.

	Galerkin	C-GP		LS-SP and C-CSP		C-ASP
		Odd p	Even p	Odd p	Even p	
L^2	$p+1$	$p-1$	p	$p+1$	p	p
H^1	p	$p-1$	p	p	p	p
H^2	$p-1$	$p-1$	$p-1$	$p-1$	$p-1$	$p-1$

Table 1: Comparisons of orders of convergence: Galerkin, C-GP, LS-SP, C-CSP and C-ASP.

Again, we assume that this problem has a unique smooth solution. Following the isogeometric paradigm, the discrete solution u_h is sought in the isogeometric space

$$u_h \in S^{p,q} = \text{span}\left\{R_k^{p,q} \circ \mathbf{F}^{-1},\, \forall k = i + (j-1)m,\, i = 1, ..., n,\, j = 1, ..., m\right\}$$

cf. (2.2), and the collocation points are the image through $\mathbf{F}(\cdot)$ of a tensor-product grid of collocation points on $[0, 1]$. The collocation method is then obtained as for the univariate case.

3.2 Greville points and C-GP

Greville points, or abscissas, for p-degree B-splines associated to a knot vector $\Xi = \{\xi_1, ..., \xi_{n+p+1}\}$ are defined as

$$\tau_i^{GP} = \frac{\xi_{i+1} + ... + \xi_{i+p}}{p}, \quad \forall i = 1, ..., n,$$

see Figure 1 for an example computed from a open uniform knot vector and degree $p = 3$ and $p = 4$. For an open knot vector the first and last Greville point coincide with the first and last knots ξ_1 and ξ_{n+p+1}. A common collocation scheme for second-order boundary value problems, as proposed in [5], uses as collocation points the $n-2$ internal Greville points. For brevity, this is denoted Collocation at Greville Points, C-GP.

In Table 1 we report the orders of convergence of C-GP and the other methods considered in this paper. The convergence rate of C-GP in L^2 norm is $p-1$ when odd degree are used and p when even degree B-Splines are used as discussed earlier (i.e. two-orders and one-order suboptimal, respectively). The error in H^1 norm converges with the same orders of the L^2 norm, and is therefore optimal for even degrees and one-order suboptimal for odd degrees. The error measured in H^2 norm is instead optimal for every degree.

3.3 Superconvergent points for the second derivative of the Galerkin solution

Following [14], we now introduce the second derivative superconvergent points for the Galerkin solution of Problem 1, which will be used to construct the collocation method.

Assume for a moment that $a_0 = a_1 = 0$, that is consider the simplified problem

$$\begin{cases} -u''(x) = f(x) & \forall x \in (0,1) \\ u(0) = u(1) = 0, \end{cases} \tag{3.6}$$

and let u_h^* be the approximated solution given by the Galerkin method based on B-splines. The points $\Psi_h = \{\psi_{h,1}, ..., \psi_{h,w}\}$ with $w \in \mathbb{N}, w > 0$, are said to be Superconvergent Points (SP) for the j-th derivative of u if

$$\Big[\sum_{\psi_{h,i} \in \Psi_h} \big[D^j(u - u_h^*)(\psi_{h,i})\big]^2 \Big]^{\frac{1}{2}} \leq Ch^{p+1-j+k}, \quad \forall i = 1 \ldots, w \tag{3.7}$$

where $k > 0$, $j \geq 0$, C is a constant, h is the meshsize of the knot vector, and p is the degree of the B-splines.

Here we are interested in the case $k = 1$ and $j = 2$. Indeed, $D^2(u - u_h^*)$ is the Galerkin residual for (3.6), for which we are (ideally) interested in assessing the zeros, which in turn we replace with superconvergence points following the idea presented in the introduction. However, finding the location of the superconvergent points is in general an open problem as well. Under the assumption that the superconvergent points are element invariant (that is, images by affine mapping of points on a reference element) their locations have been estimated in [14] and are reported in Table 2 for a reference element $[-1, 1]$. The same points are estimated in [1] under a similar periodicity assumption. Both assumptions do not hold true in many cases of interest. The superconvergence theory of [26] is instead based on a mesh symmetry assumption, which however does not hold true for elements close to the boundary.

Following [1] and [14], since we do not have access to the "true" superconvergent points, we use the points in Table 2, linearly mapped to the generic element, as "surrogate" superconvergent points in one-dimension. For easiness of exposition, we refer to them throughout this paper as "superconvergent points", although this might not be technically true. How good is the approximation in practice? Figures 2 and 3 show $D^2(u - u_h^*)$ for equation (3.6) with $f(x) = \sin(\pi x)$, over a mesh with 10 and 20 elements and $p = 3, \ldots, 7$, as well as the "surrogate" superconvergent points for each degree of approximation: for odd degrees, a non-negligible discrepancy is evident at the boundary of the interval, and for even degrees this occurs also at the middle of the interval. Figure 4 is a zoom of the first element in Figure 3.

For completeness, Figure 5 shows the residual for the Periodic Problem 2 with $a_0 = a_1 = 1$, and $f(x) = (1 + 4\pi^2)\sin(\pi x) + 2\pi \cos(2\pi x)$ over a mesh with 10 elements and $p = 3, \ldots, 7$, as well as the "surrogate" superconvergent points for each degree of approximation. In this case, the mismatch between the zeros of the residual and the "surrogate" superconvergent points is higher in correspondence of a smaller residual. Note that the residual is not periodic at the element scale.

For multi-dimensional problems on a NURBS single-patch geometry, the superconvergent points can be obtained by further mapping the tensor product of one-dimensional superconvergent points through the geometry map $\mathbf{F}$ in the physical domain. Clearly, the same considerations of the one-dimensional case are valid.

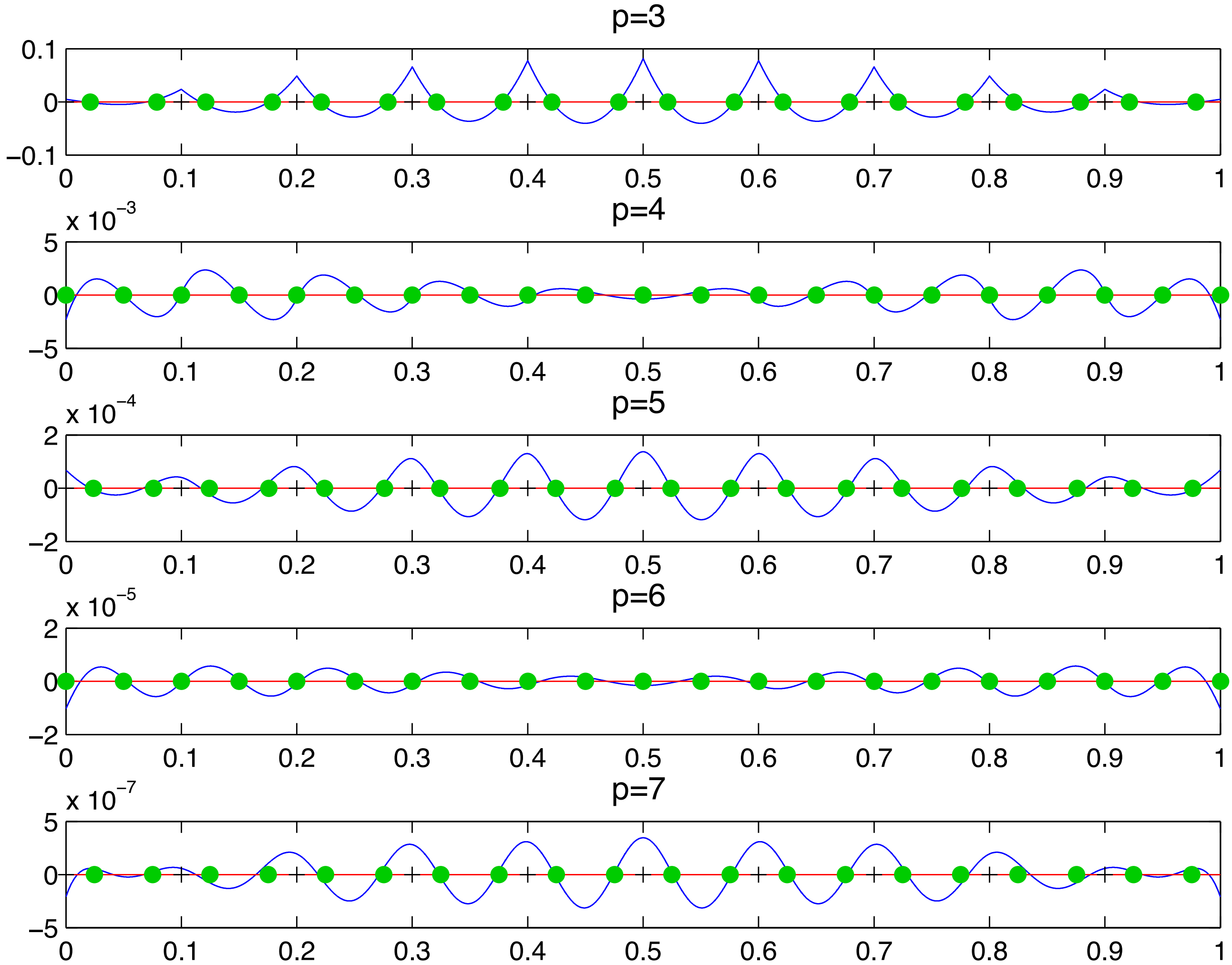


Figure 2: Plot of $D^2(u - u_h^*)$, equivalent to the residual of problem (3.6), and "surrogate" superconvergent points (green dots), on a mesh with 10 elements.

Degree	Second derivative SP
p=3	$\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$
p=4	-1,0,1
p=5	$\pm\frac{\sqrt{225-30\sqrt{30}}}{15}$
p=6	-1,0,1
p=7	± 0.504918567512

Table 2: On the reference element $[-1, 1]$, location of superconvergent points for the second derivative (from [14]).

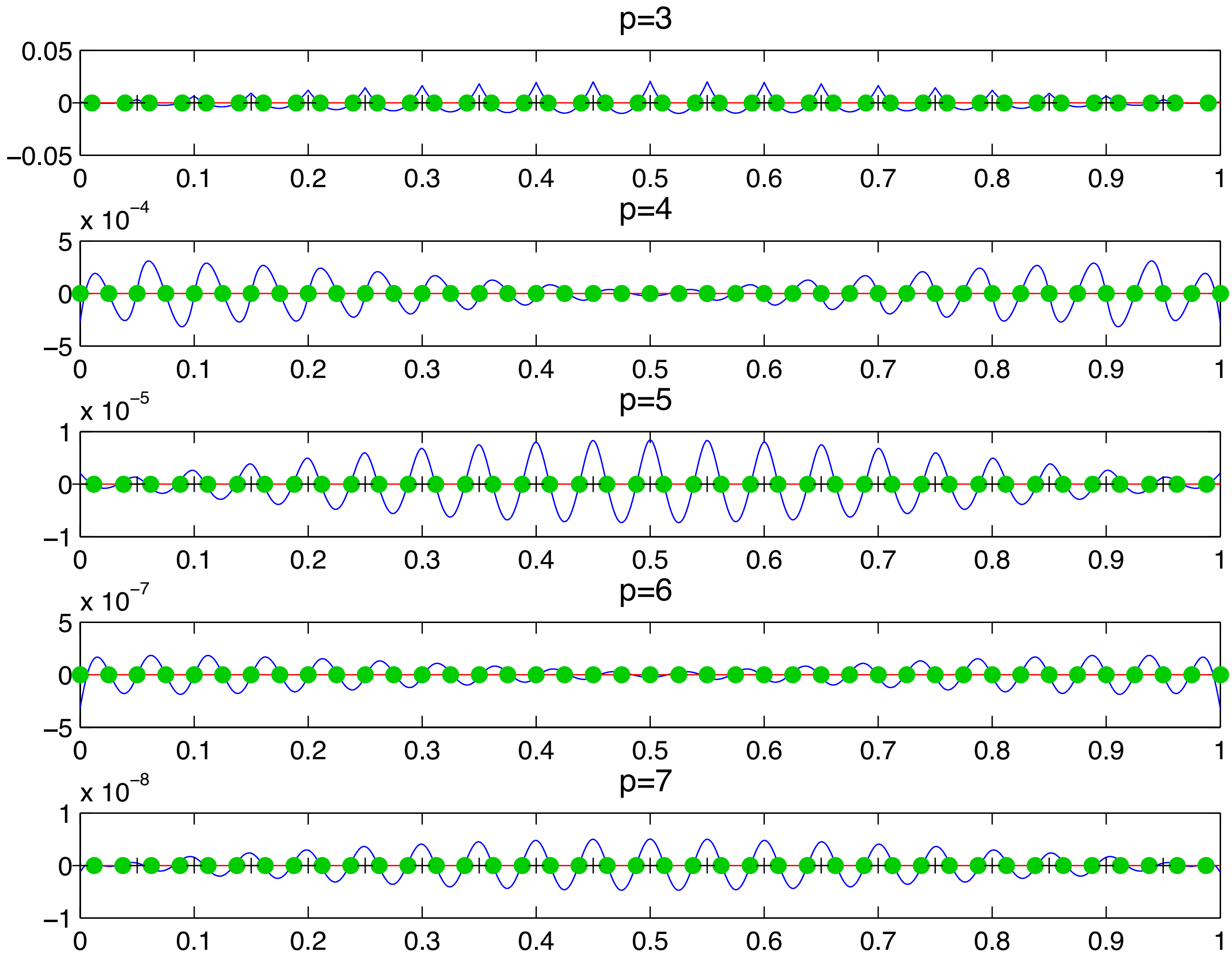


Figure 3: Plot of $D^2(u - u_h^*)$, equivalent to the residual of problem (3.6), and "surrogate" superconvergent points (green dots), on a mesh with 20 elements.

3.4 Least-Squares at Superconvergent Points (LS-SP)

As already mentioned, the Least-Squares at Superconvergent Points method (LS-SP) has been introduced by [1]. In this method all the superconvergent points are used as collocation points. As it can be seen in Table 2, there are at least two superconvergent points per element; if we take all of them as collocation points, we obtain an overdetermined system of equations if the numbers of element is large enough: such linear system is then solved in least square sense, leading to a method which is not strictly a collocation method. The order of convergence of the method as measured in numerical tests is reported in Table 1: note that it is optimal for odd degrees and one-order sub-optimal in L^2 for even degrees, while it is optimal regardless of the parity of p in H^1 and H^2 norm.

Figure 6 shows the superconvergent points for $p = 3, \ldots, 7$ on a knot vector with 10 elements. Observe that the same least-square formulation can accommodate for both Dirichlet problems (i.e., open knot vectors) and periodic problems.

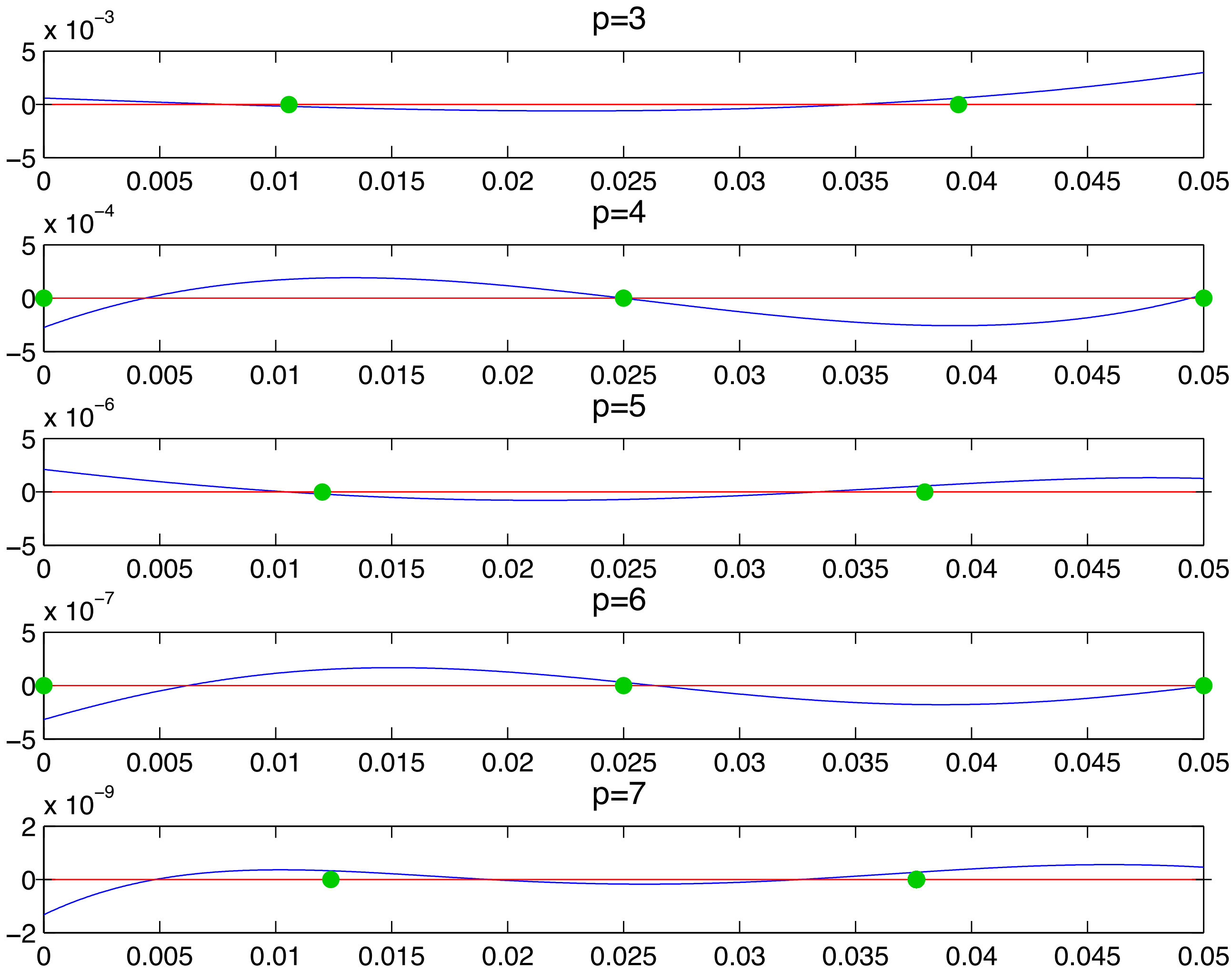


Figure 4: Zoom on the first element of Figure 3.

3.5 Collocation at Alternating Superconvergent Points (C-ASP)

C-ASP has been introduced in [14], as already discussed in the introduction, and can be seen as a collocation method derived from LS-SP, where a subset of superconvergent points with cardinality equal to the number of degrees-of-freedom to be determined is employed as set of collocation points.

To this end, the authors of [14] propose to select a subset of the superconvergent points in such a way that every element of the knot span contains at least one collocation point; note that this roughly means considering every other superconvergent point, hence the name we give to the method. Because we need to select as many collocation points as degrees of freedom, the easiest case is when one considers the periodic Problem 2, for which the number of elements is identical to the number of degrees-of-freedom, so that exactly one superconvergent point per element is selected, see Figure 7 (this case is not considered in [14]). Note that for even p one possibility is then to select the midpoint of each element, i.e., the Greville points for the uniform knot vector, see Figure 7b. For the Dirichlet Problem 1, one needs instead to select $n_{el} + p - 2$ collocation points on a mesh of n_{el} elements. To this end, an ad-hoc algorithm is presented in [14] that selects suitable superconvergence points in the internal part of the domain, and "blends them" with Greville points on the elements close to the boundary, as can be seen in Figure 8. Note that other choices for the elements close to the boundary can be envisaged,

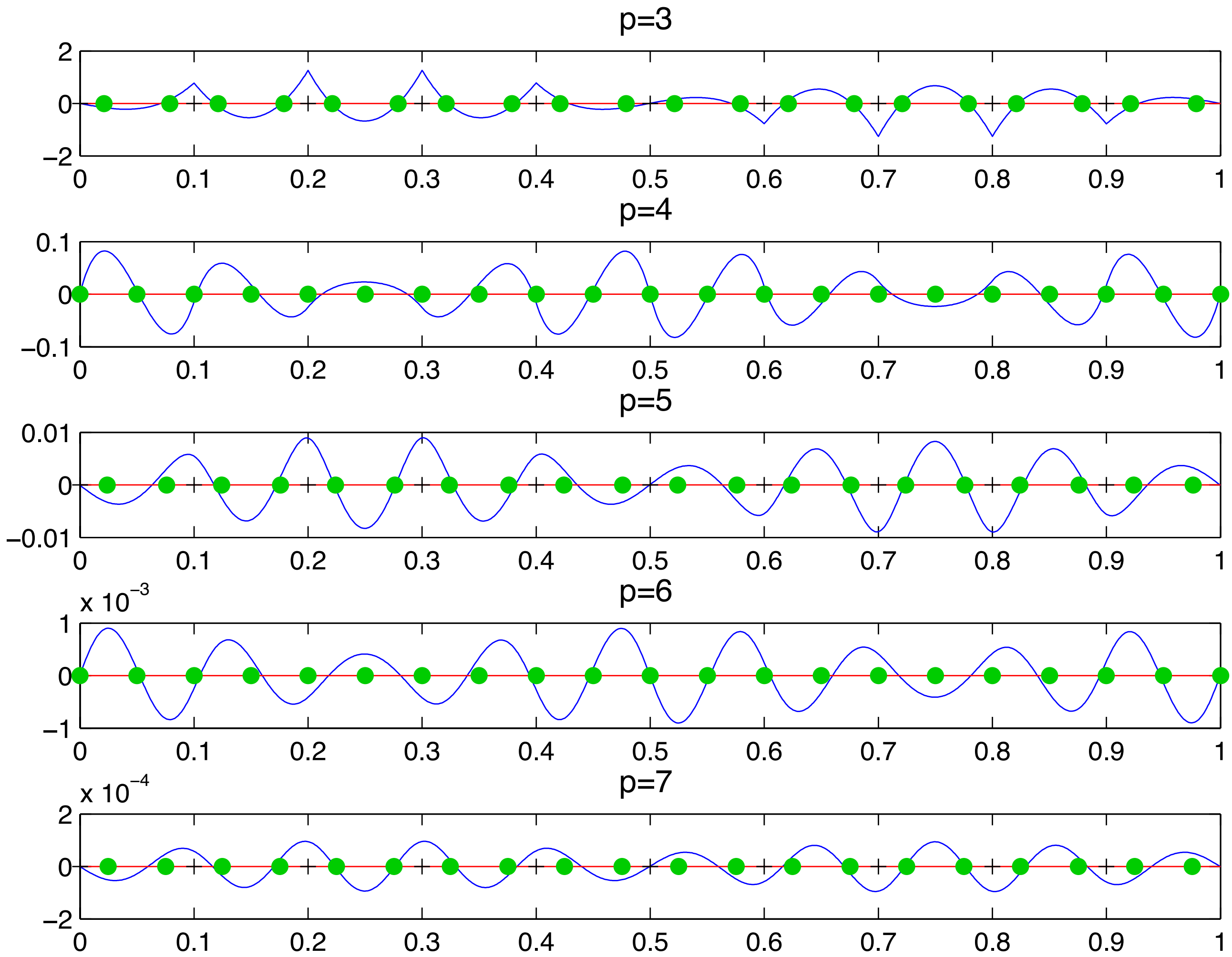


Figure 5: Residuals of periodic problem with 10 elements.

which however do not affect the convergence order of the method, see [20].

The convergence orders of C-ASP assessed numerically by [1] are also reported in Table 1. Note in particular that the L^2 order of convergence for C-ASP is p regardless of the parity of p, i.e., one-order suboptimal, while the H^1 and H^2 orders of convergence are optimal, again regardless of the parity of p.

3.6 Collocation on Clustered Superconvergent Points (C-CSP)

We now describe a new choice of collocation points among the superconvergent points, alternative to C-ASP, which we name Collocation on Clustered Superconvergent Points (C-CSP).

To understand our approach, we describe it first in the simplest setting, i.e., the periodic Problem 2 with even number of elements and odd degree p. We look for a periodic distribution of collocation points but keeping the symmetry at the element level. This can be achieved selecting two superconvergent points in an element and then skipping the following one, as depicted in Figure 9. Surprisingly, the order of convergence of C-CSP in this case is optimal, cf. the numerical results in Section 4, Figure 14. For even degrees, we have experimented different selections of sets of superconvergent points, preserving periodicity and some local symmetry, two of which are depicted in Figure 10 (observe that with the first one we end up with Greville points again). In all

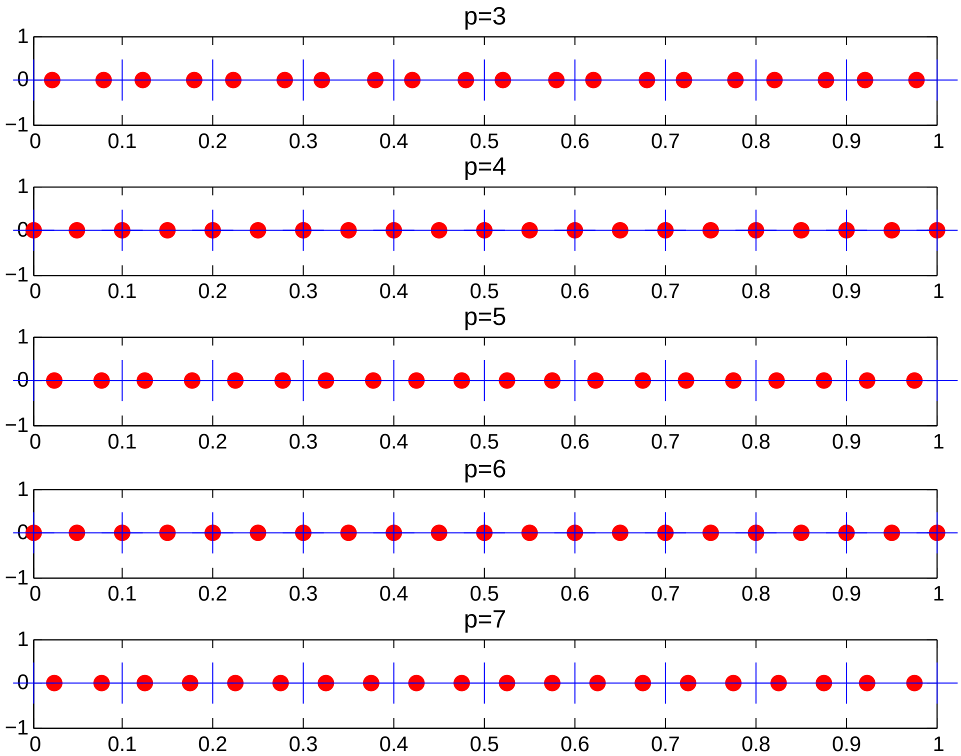
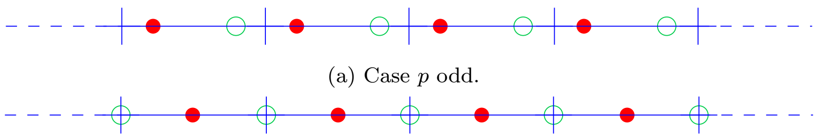


Figure 6: Superconvergent points for $p = 3, \ldots, 7$ on a knot vector with 10 elements.



(a) Case p odd.

(b) Case p even.

Figure 7: Example of C-ASP points for the periodic Problem 2. The collocation points are marked with full red dots, while the remaining superconvergent points are displayed with green circles. In this case, C-ASP and C-GP coincide for even degrees.

cases, we have measured numerically one-order suboptimal convergence in L^2, i.e., we do not see improvements with respect to C-GP, LS-SP or C-ASP, see the numerical results in Section 4, Figure 15. At this point, only the odd-degree C-CSP seems to deserve further interest, and we will restrict to this case in the remaining of the paper. How to use efficiently the superconvergent points in an even degree splines collocation scheme remains an open problem.

The next step is to extend (odd-degree) C-CSP to the open knot vector to solve the Dirichlet Problem 1. To this end, we need to include additional points, which are taken among the other superconvergent points populating the elements close to the boundary, and trying to preserve symmetry, see Figures 11 and 12. Note that when the number of elements is even the procedure just described will not yield a globally symmetric distribution of collocation points, cf. Figure 12. We can however restore symmetry with a little modification of the collocation approach: we add one (or a few) points to the collocation set to restore symmetry of the collocation scheme, and average the equations corresponding to the points located at the center of the domain

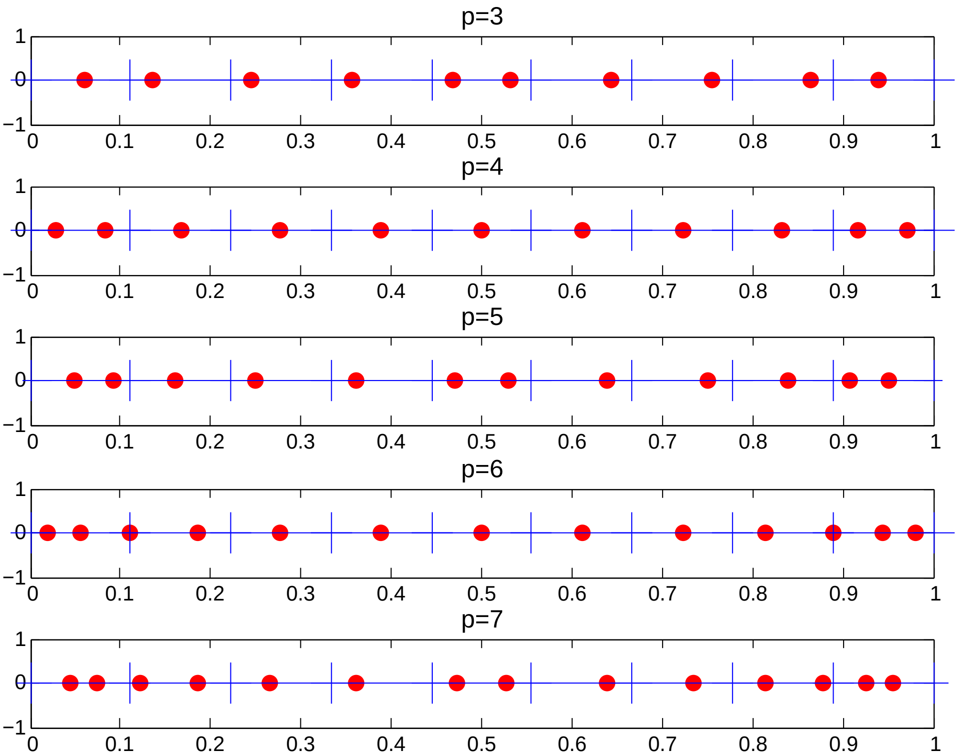


Figure 8: Example of C-ASP points for the Dirichlet problem 1 over a knot vector with 9 elements. The points adjacent to the boundary are obtained according to Algorithm 1 of [14].

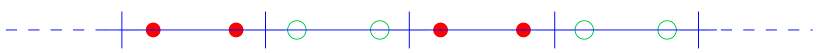


Figure 9: Periodic C-CSP stencil for odd degree: the collocation points are marked with full red dots, while the remaining superconvergent points are displayed with green circles.

in order to match the number of unknown. This procedure is depicted in Figure 13 for $p = 3$.

The order of convergence of C-CSP on regular meshes, reported in Table 1, is the same of LS-SP, i.e., optimal for odd degrees. As already mentioned, all our attempts to extend it to even degree splines has produced one order suboptimal convergence. These convergence rates have been measured by running the numerical benchmarks detailed in Section 4 (also covering the symmetric variant of Figure 12).

4 Numerical tests

This section is devoted to the numerical benchmarking of the new C-CSP method, and its comparison to the other approaches recalled in Section 3. For conciseness, we do not show convergence results in L^∞ norm, which we found to be identical to the ones in L^2 norm in each of the tests reported below.

We begin by testing C-CSP on the periodic Problem 2, with $a_0 = a_1 = 1$ and with $f(x) = (1+4\pi^2)\sin(2\pi x) + 2\pi\cos(2\pi x)$, whose solution is $u(x) = \sin(2\pi x)$. As previously discussed, this is the only test for which we present results for even degrees p: we see from the plots in Figures 14 and 15 that the orders of convergence for the L^2 norm of the error are optimal, i.e. equal to $p+1$, for odd values of p, while for even p the measured convergence

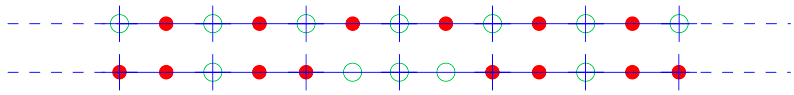


Figure 10: Attempts of C-CSP stencil for even degree: the collocation points are marked with full red dots, while the remaining superconvergent points are displayed with green circles. The construction at the top leads to Greville points, while the one at the bottom yields symmetry at a macro-element level.

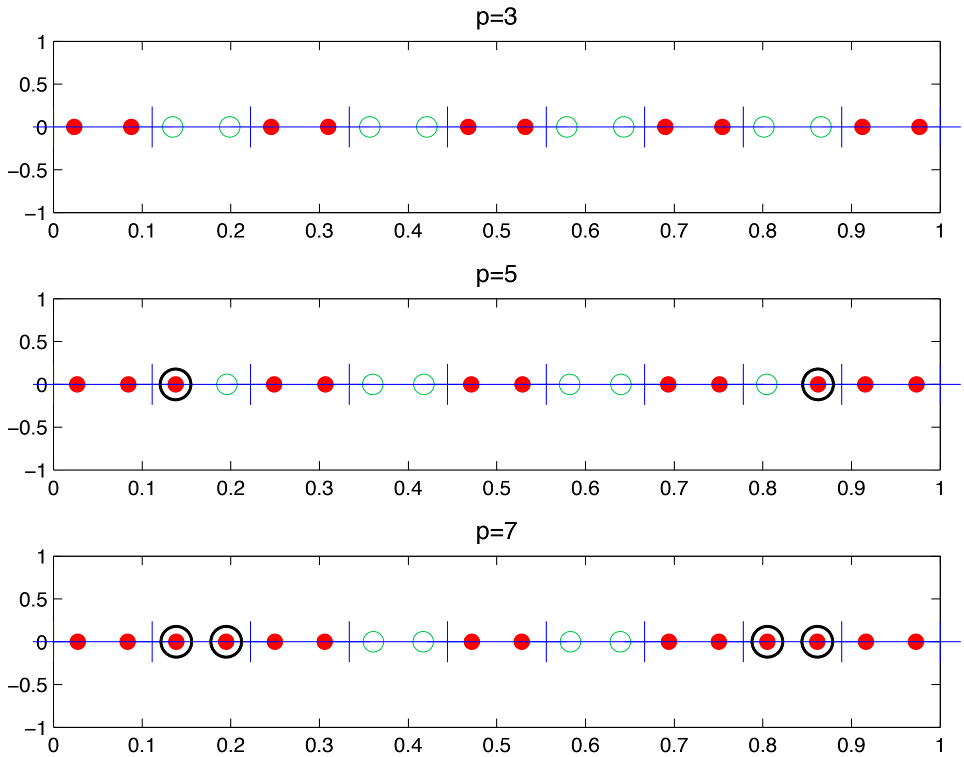


Figure 11: C-CSP points for a Dirichlet problem solved on a mesh with 9 elements (*odd number of elements*, leading to a symmetric set of point): the collocation points are marked with full red dots, while the remaining superconvergent points are displayed with green circles. Black dots represent the points added with respect to the periodic stencil.

rate is only p, i.e. one-order suboptimal.

A natural question arises: why can't we achieve optimal convergence when even degrees B-splines are considered? The answer is not yet clear. As we explained in the previous sections, the rationale behind C-CSP, as well as LS-SP and C-ASP, is to try to obtain the same solution delivered by the Galerkin method by imposing the collocation residual to be zero at the superconvergent points, which are supposedly close to the true zeros of the Galerkin residual. However, as we discussed in Section 3.3, we do not have access to the precise location of the superconvergent points, and instead we use "surrogate" superconvergent points that do not approximate well the zeros of the Galerkin residual everywhere in the domain. We do not see however any qualitative difference between the odd and even case other than in the central element (although we did not perform a quantitative analysis of this issue). Furthermore, it is not clear why the C-ASP selection results worse of the C-CSP one: in other words, the points that would be selected by the C-ASP seem as good as those that would be selected by

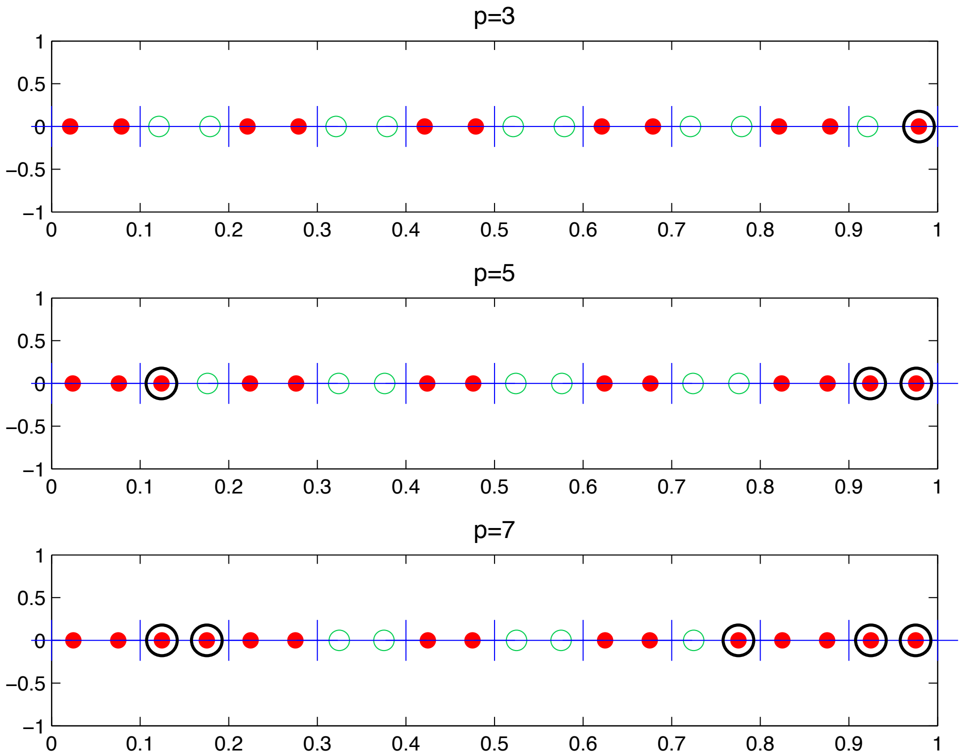


Figure 12: C-CSP points for a Dirichlet problem solved on a mesh with 10 elements (*even number of elements*, leading to a non-symmetric set of point): the collocation points are marked with full red dots, while the remaining superconvergent points are displayed with green circles. Black dots represent the points added with respect to the periodic stencil.

the C-CSP as for what concerns being close to the zeros of the Galerkin residual.

We continue by testing the C-CSP method on the Dirichlet Problem 1 with $a_0 = a_1 = 0$ and $f(x) = \pi^2 \sin(\pi x)$, whose exact solution is $u(x) = \sin(\pi x)$, and show the corresponding results in Figure 16. As in the previous case, the order of convergence is $p+1$ in L^2 norm and it is p in H^1 norm. In order to compare the four methods we presented (C-CSP, ASP, LS-SP and Greville collocation) against the Galerkin solver, we show in Figure 17 a comparison of the convergence of the L^2-error obtained when solving the Dirichlet problem above with B-splines of degree $p = 3$. The plot highlights that C-CSP, although converging with optimal order, shows an accuracy of one order of magnitude larger than Galerkin, while LS-SP converges essentially to the same solution of the Galerkin method. It should be observed however that the computational cost of LS-SP is significantly higher than C-CSP, not only because of the number of points where the residual needs to be evaluated (about 2^d times more that C-CSP in d dimensions) but also for the the higher condition number of the resulting system of linear equations.

We also investigate the robustness of the method with respect to perturbations of the knot vector. To this end, we start from an open knot vector with equispaced knots in the interior of $[0, 1]$, we perturb it by randomly chosen quantities, i.e. we replace each internal knot ξ_i by $\tilde{\xi}_i = \xi_i + \frac{1}{10 n_{el}} X$, where X is a random number $X \in [-1, 1]$ (the resulting knot vectors are then said quasi-uniform; observe that the scaling factor $\frac{1}{10 n_{el}}$ prevents knot clashes) and then we place the superconvergent points in each of the resulting elements.

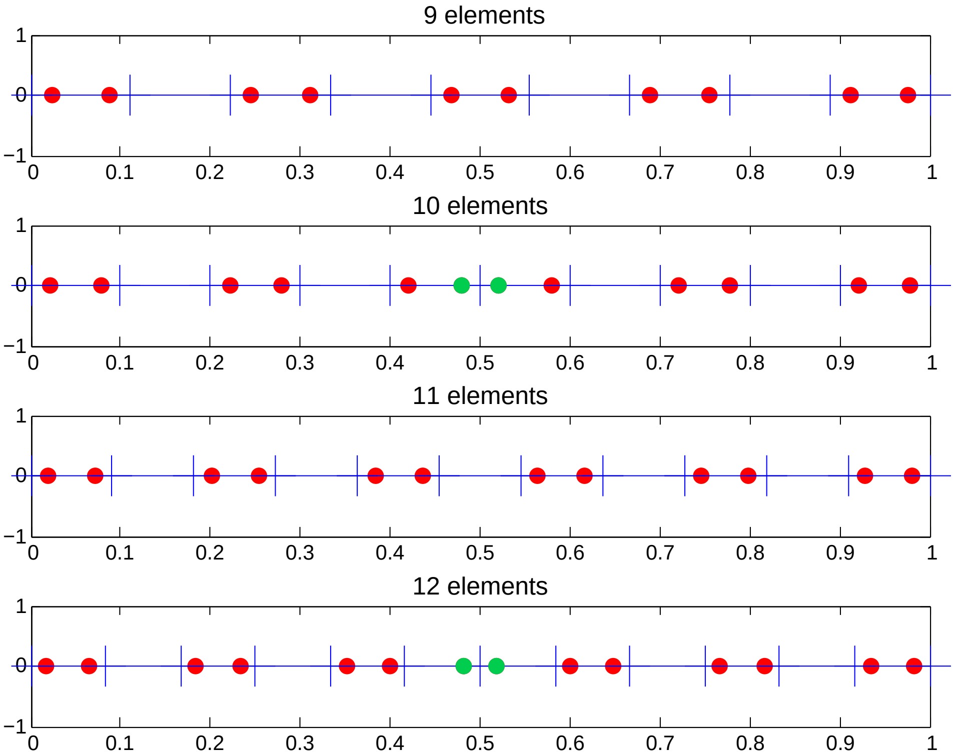


Figure 13: C-CSP symmetric-variant points for a Dirichlet problem with $p = 3$: the collocation points are marked with red dots, while the superconvergent points whose equations have to be averaged are displayed with green dots.

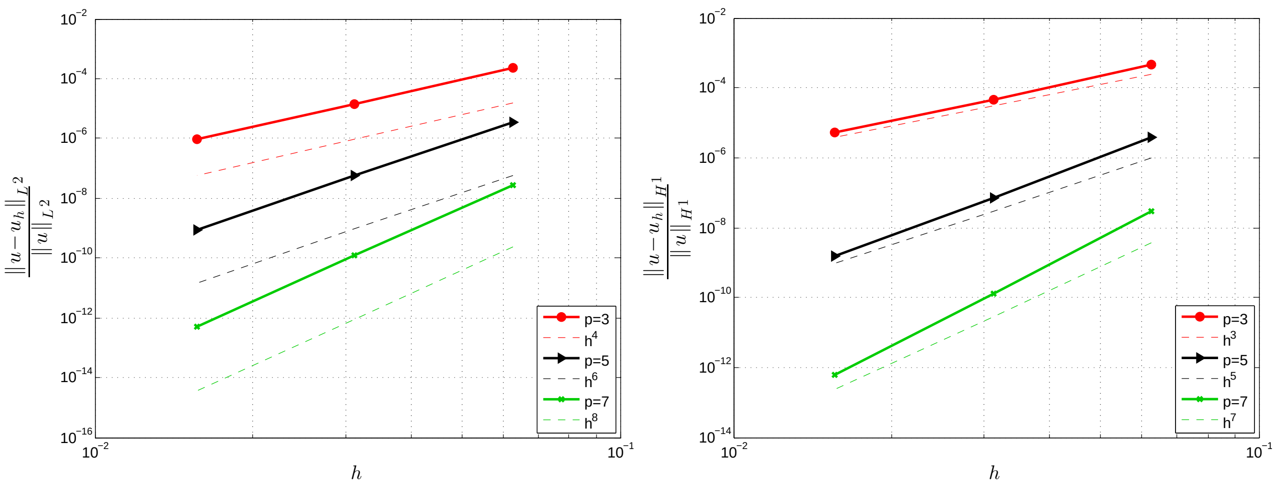


Figure 14: L^2 and H^1 error plot: C-CSP periodic problem (odd p)

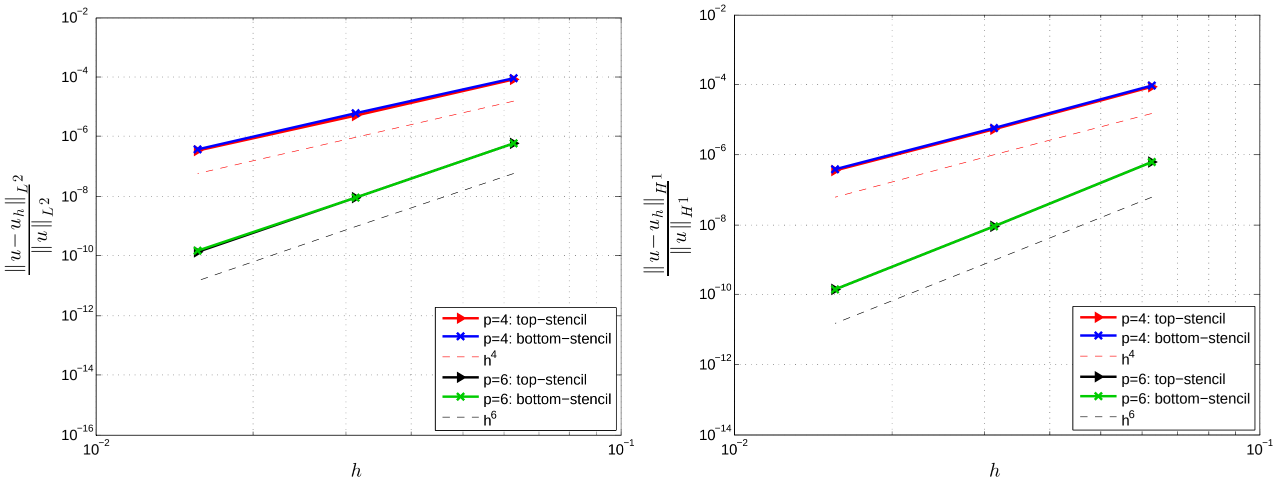


Figure 15: L^2 and H^1 error plot for C-CSP periodic problem (even p), with the stencils depicted in Figure 10.

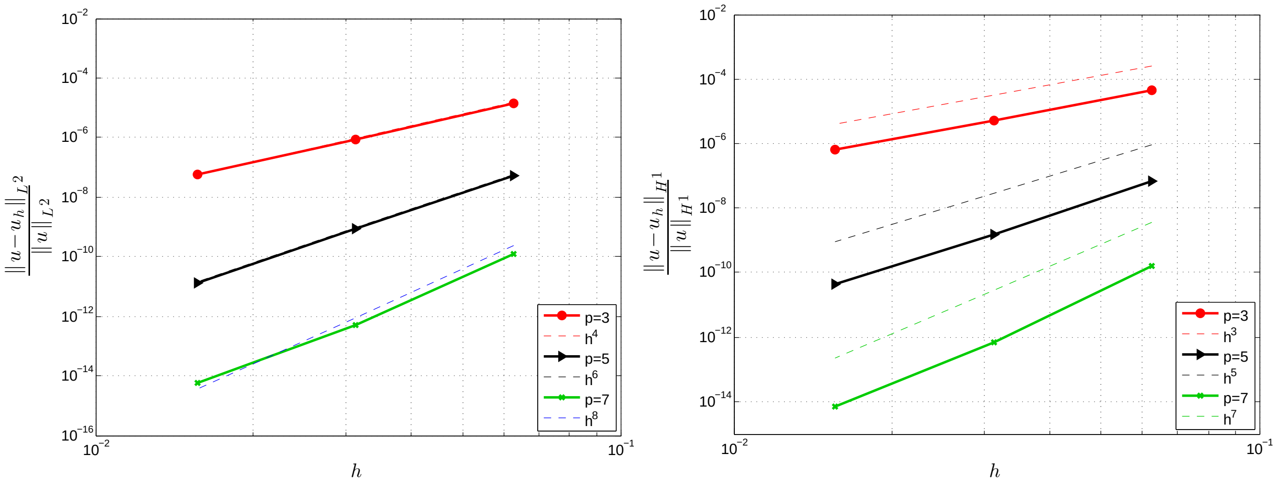


Figure 16: L^2 and H^1 error plot: C-CSP Dirichlet problem.

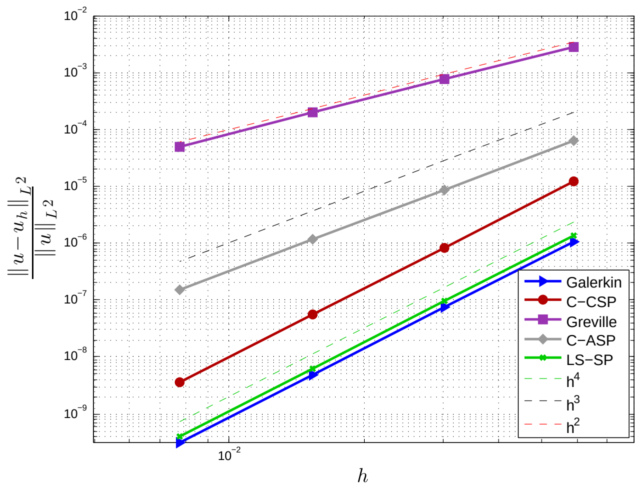
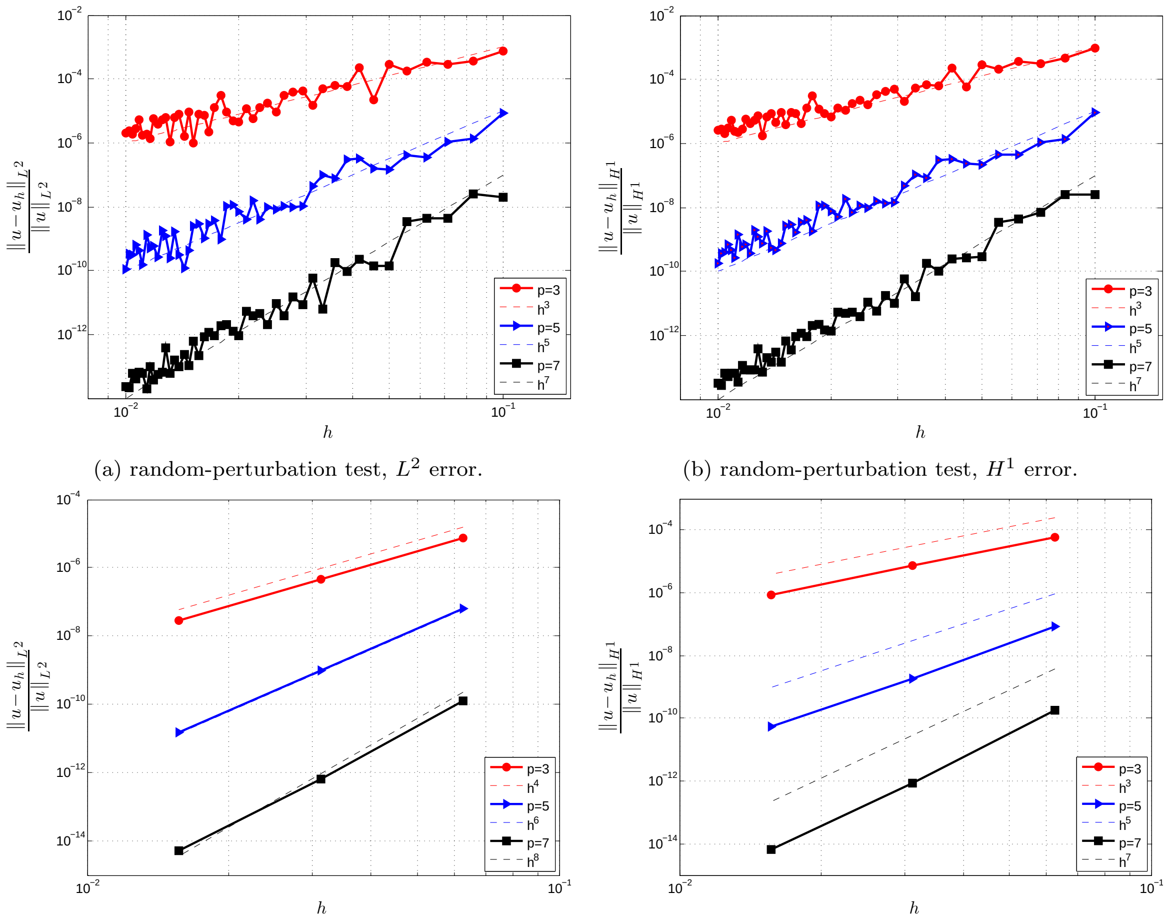


Figure 17: Comparison of convergence of L^2 error norms for the Dirichlet problem for different methods.



(a) random-perturbation test, L^2 error.

(b) random-perturbation test, H^1 error.

(c) advection-reaction test, L^2 error.

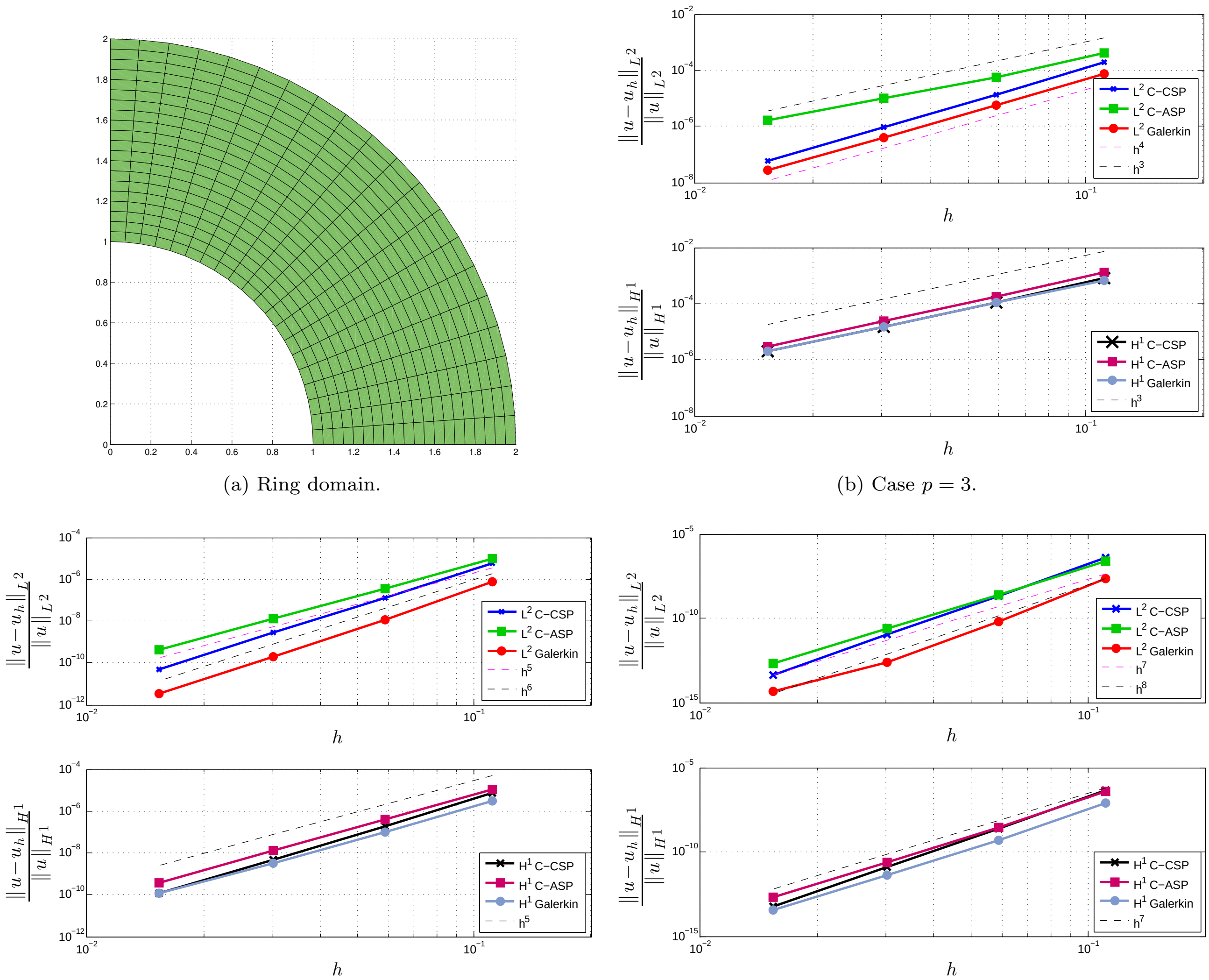
(d) advection-reaction test, H^1 error.

Figure 18: Robustness test for C-CSP with respect to perturbation of the knot vectors (plots 18a and 18b) and changes of the differential operator (plots 18c and 18d).

The error plots are shown in Figure 18a and 18b. We note that we loose the optimal rates of convergence we observed in the previous tests: the order of convergence is p for both the L^2 and H^1 error norm, i.e., optimal for the H^1 error norm and one-order suboptimal for the L^2 one.

We also verify the influence of the differential operator, by considering non-null a_0 and a_1 in the Dirichlet Problem 1. We performed several tests and the results obtained were identical; therefore, we report here only one representative example. In detail, we consider $a_1(x) = x$, $a_0 = 1$ and $f(x) = x(e^x \sin(\pi x) + \pi e^x \cos(\pi x)) - 2\pi e^x \cos(\pi x) + \pi^2 e^x \sin(\pi x)$, whose exact solution is $u(x) = \sin(\pi x)e^x$. The results of the test we performed are shown in Figure 18c and 18d: the order of convergence is still optimal: $p+1$ for the L^2 error norm, and p for the H^1 norm. We can then conclude that the C-CSP method seems to be robust with respect the form of the elliptic operator.

Finally, we present two examples of two-dimensional Dirichlet Problem 3 solved by C-CSP. In the first one we consider as computational domain Ω the quarter of annulus in Figure 19a (for which NURBS functions have to be employed), and $f(x, y)$ is chosen such that the exact solution is $u(x, y) = -(x^2 + y^2 - 1)(x^2 + y^2 - 4)xy^2$. In Figures 19b, 19c and 19d we show the convergence plots of the L^2 and H^1 errors for the C-CSP and Galerkin methods for odd degree NURBS $p = 3, 5, 7$. The observed orders of convergence are as expected: optimal order



(a) Ring domain. (b) Case $p = 3$.

(c) Case $p = 5$. (d) Case $p = 7$.

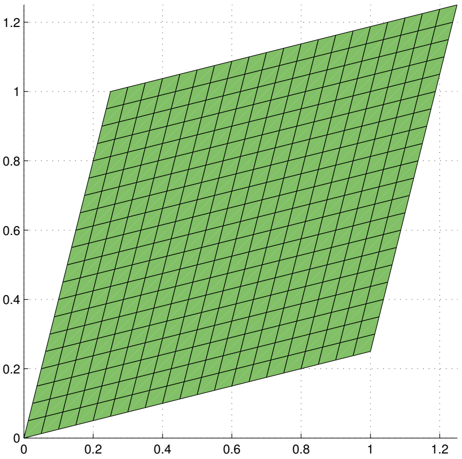
Figure 19: Ring domain and L^2 and H^1 convergence of approximations with $p = 3, 5, 7$.

in L^2 norm and H^1 norm. We remark that for $p = 3$ the H^1 norms of the corresponding errors obtained by the Galerkin and C-CSP methods are very close, as can be seen in Figure 19b.

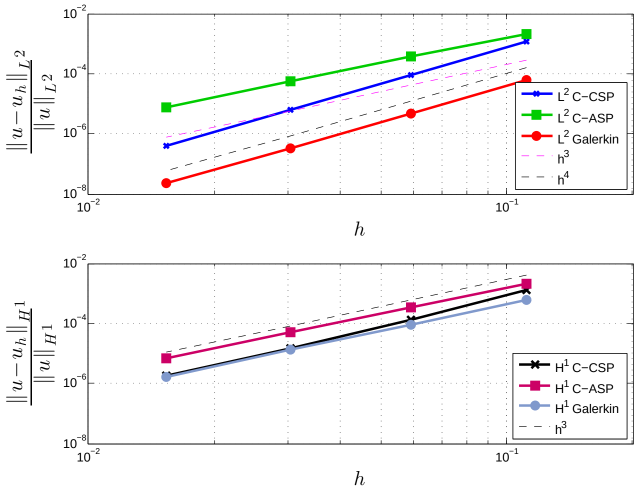
In the second two-dimensional example, we let Ω be the rhombus with vertices $(0, 0)$, $(\frac{1}{4}, 1)$, $(1, \frac{1}{4})$ and $(\frac{5}{4}, \frac{5}{4})$, represented in Figure 20a (its parametrization is bilinear but not orthogonal, as in the previous example), and f is such that the exact solution is $u(x, y) = \sin\left(\frac{4}{15}\pi(y - 4x)\right)\sin\left(\frac{16}{15}\pi\left(\frac{x}{4} - y\right)\right)(x^3 + y^3)$. The corresponding errors are shown in Figures 20b, 20c and 20d, and the same observations as before hold. Note however that the gap between the C-CSP and the Galerkin solution is larger than in the previous example, especially for the L^2 error. Moreover, for $p = 5$ and $p = 7$ the convergence is still in its preasymptotic regime.

5 Conclusions

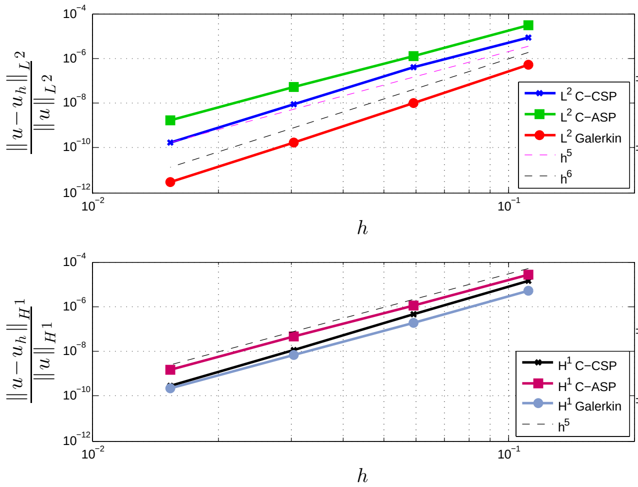
In this paper we have proposed an isogeometric collocation method based on the superconvergent Galerkin points. Our method uses as collocation points a subset of the superconvergent points, similarly to what proposed in [14]. Our guiding criterion is however different, and consists in picking up clusters of points, in such a way to



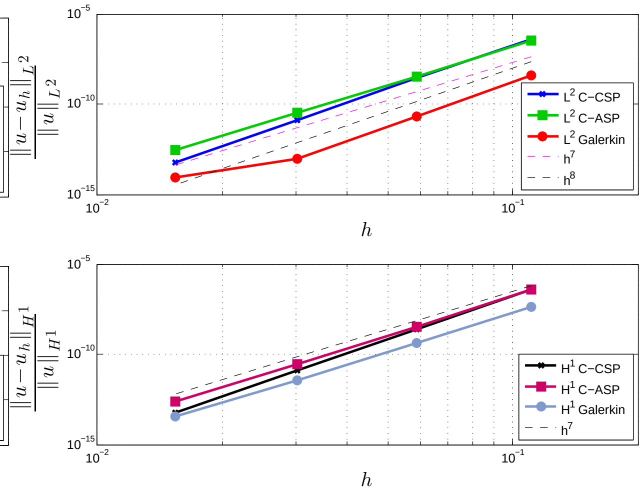
(a) Rhombus domain



(b) Case $p = 3$.



(c) Case $p = 5$.



(d) Case $p = 7$.

Figure 20: Rhombus domain and L^2 and H^1 convergence of approximations with $p = 3, 5, 7$.

preserve symmetry at the element level. This choice allows to recover optimal convergence rates for odd-degrees splines/NURBS, without "oversampling" the domain as in least-square approach proposed by [1]. Moreover, the order of convergence of the L^∞ norm of the error is the same of L^2-norm in all experiments we have performed (not shown in the paper for the sake of brevity).

The preliminary numerical campaign on one- and two-dimensional tests that we performed suggests that the method is robust with respect to isogeometric mapping of the domain, while perturbations of the knot vector may reduce the accuracy of the method. A rigorous mathematical explanation for the convergence behavior observed for the proposed method, and for the other collocation methods based on the Galerkin superconvergent points, is not available yet and will be the target of our future efforts.

Acknowledgements

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